



University of  
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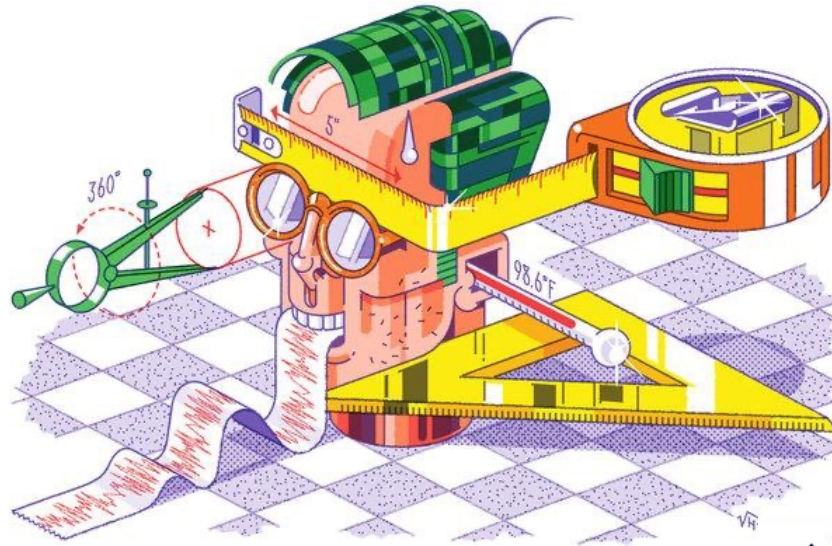
Department of Geography



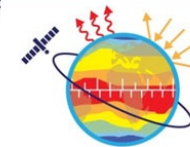
# Basics of Uncertainty of Measurement

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Metrology for Earth  
Observation and Climate



## Basics of Uncertainty of Measurement

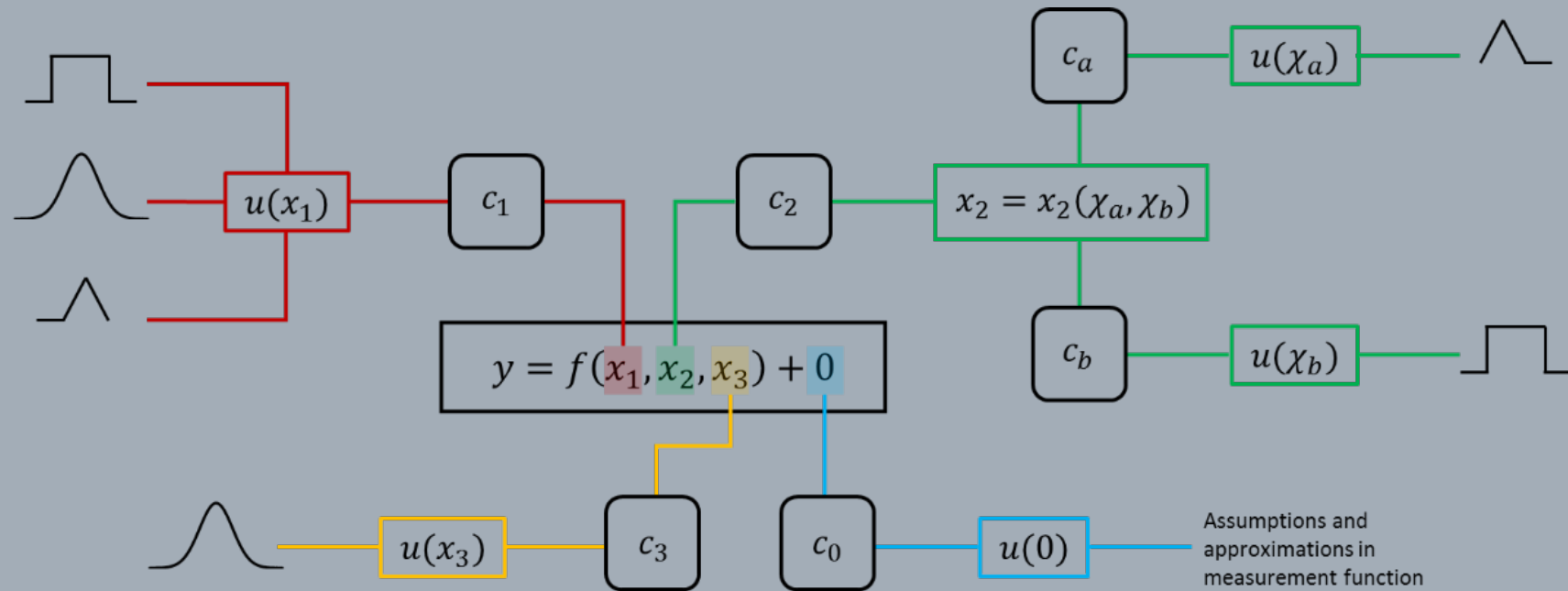


- Introduce the subject of measurement uncertainty and *why it matters*
- Understand when to use statistical analysis of a set of measurements and when to use other kinds of information about the measurement process in order to estimate measurement uncertainties
- Know the established rules for how to calculate an overall estimate of uncertainty from these individual pieces of information



## Basics of Uncertainty of Measurement

*End goal... measurement-function centred analysis of uncertainty and graphical representation in the form of an uncertainty analysis tree*





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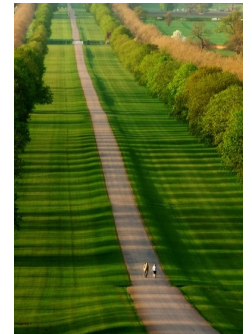
1. What is a measurement (and what isn't)
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# **1. What is a measurement (and what isn't)**

## 1.1 What is a *measurement*?

➔ A measurement **tells us about a property of something.**



➔ A measurement **gives a number to that property.**



➔ Measurements are *a/ways* made **using an instrument of some kind.**



➔ The **result of a measurement** is generally in 2 parts: **a number and a unit of measurement.**

## 1.2 What is *NOT* a measurement?

➡ Comparing 2 pieces of string to see which is longer.



➡ Counting is not normally viewed as a measurement.



➡ **A test is not a measurement:** tests normally lead to a 'yes/no' answer or a 'pass/fail' result (however, measurements may be part of the process leading up to a test result).





## **2. Units of measurement and metrological traceability**



## 2.1 About units of measurement...



Use the **International System of Units (SI)** which **defines seven units of measure as a basic set** from which all other SI units can be derived.

The **SI base units form a set of mutually independent dimensions** as required by dimensional analysis commonly employed in science and technology





## 2.2 What is metrological traceability?

“The property of a measurement result whereby the result can be related to a reference through a documented unbroken chain of calibrations, each contributing to the measurement uncertainty.”

JCGM 200, *International Vocabulary of Metrology. Basic and General Concepts and Associated Terms (VIM)*, 3<sup>rd</sup> ed., BIPM: Sèvres, France. [http://www.bipm.org/utils/common/documents/jcgm/JCGM\\_200\\_2012.pdf](http://www.bipm.org/utils/common/documents/jcgm/JCGM_200_2012.pdf) accessed in September 2019.

## 2.3 Traceability? I love it when a plan comes together

### Mars Climate Orbiter

➡ Total mission cost: **\$327.6 million**

➡ Spacecraft lost due to ground-based computer software which produced output in non-SI units of pound-force seconds (lbf·s) instead of the SI units of newton-seconds (N·s) specified in the contract between NASA and Lockheed

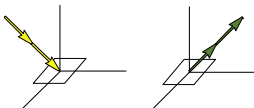
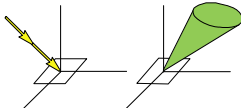
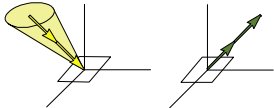
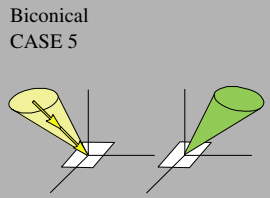
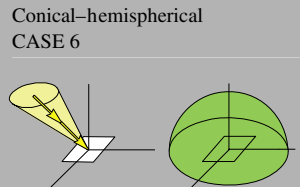
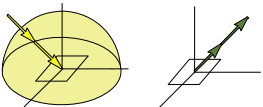
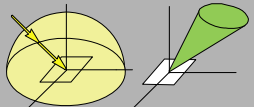
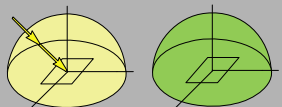
➡ The discrepancy between calculated and measured position had been noticed earlier by at least two navigators, whose concerns were dismissed because they "did not follow the rules about filling out [the] form to document their concerns"<sup>1</sup>.  
<sup>1</sup>Oberg, James (December 1, 1999). "Why the Mars Probe went off course". *IEEE Spectrum*. IEEE.

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## A word about units in remote sensing: Definitions are important and ambiguities are a source of uncertainty

Relation of incoming and reflected radiance terminology used to describe reflectance quantities

| Incoming/Reflected   | Directional                                                                                                                | Conical                                                                                                                 | Hemispherical                                                                                                              |
|----------------------|----------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------|
| <i>Directional</i>   | Bidirectional<br>CASE 1<br>               | Directional–conical<br>CASE 2<br>     | Directional–hemispherical<br>CASE 3<br> |
| <i>Conical</i>       | Conical–directional<br>CASE 4<br>        | Biconical<br>CASE 5<br>              | Conical–hemispherical<br>CASE 6<br>    |
| <i>Hemispherical</i> | Hemispherical–directional<br>CASE 7<br> | Hemispherical–conical<br>CASE 8<br> | Bihemispherical<br>CASE 9<br>         |



### **3. The measurement process: from the sources of inaccuracy to the effects of inaccuracy**



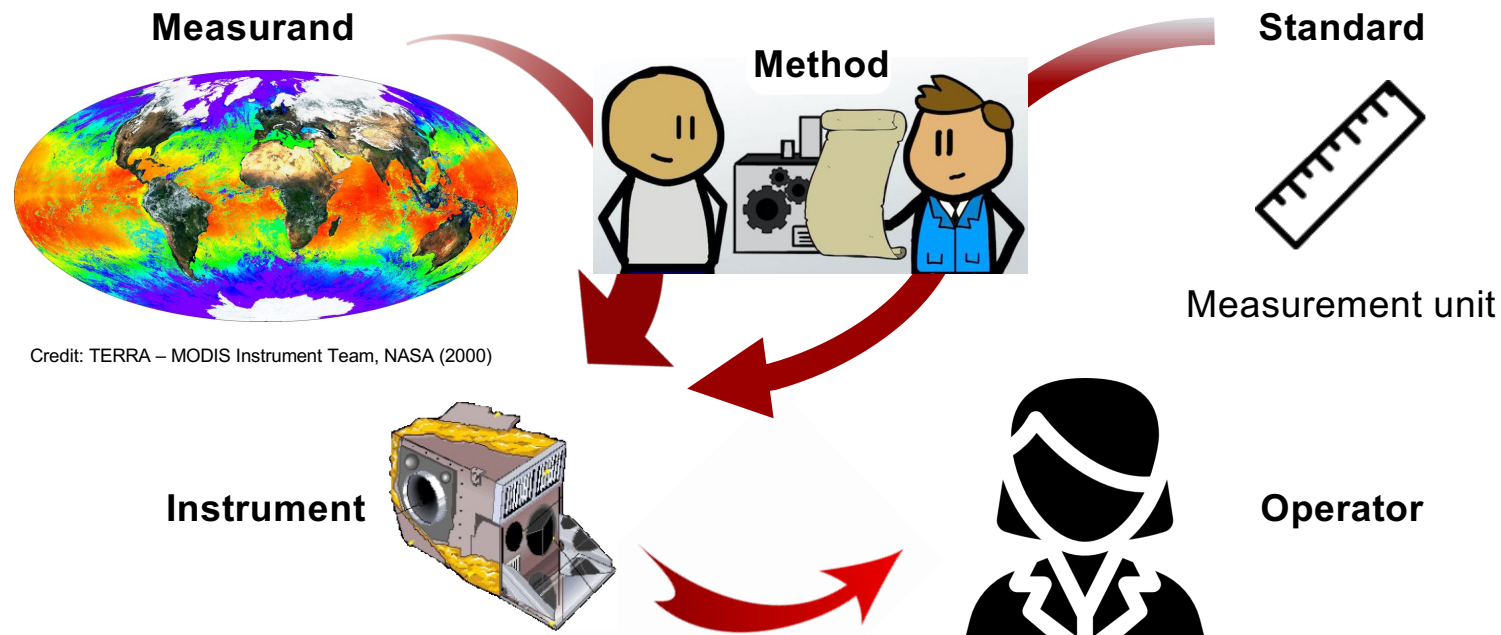
## 3.1 The measurement process

“The measurement of a quantity is generally defined as the **quantitative comparison** of this same quantity with another one, which is homogeneous with the measured one, and is considered as the **measurement unit**.”

Prof. Alessandro Ferrero, Electrical and Electronic Measurements, Polytechnic University of Milan

## 3.1 The measurement process

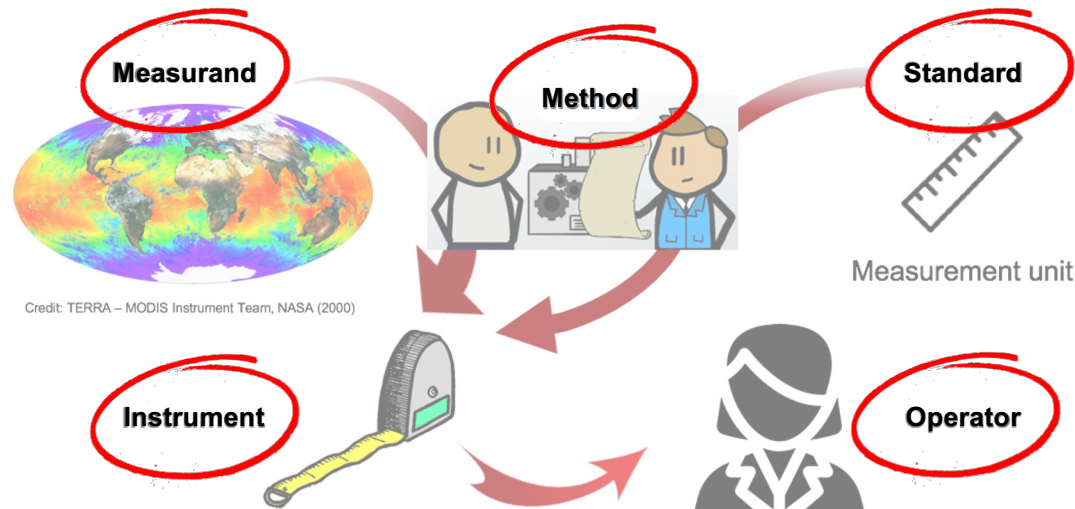
The measurement process can be described by the interaction of **5 agents**



## 3.2 The measurement process: sources of inaccuracy

Can we get the true value of the measurand as the result of a measurement?

**Hell no** → All agents taking part in the measurement process concur to make the measurement result different from the “true” value



## 3.2.1 The measurement process: sources of inaccuracy

### The measurand

➔ **Knowledge** of the measurand is very often **incomplete**



➔ **Mathematical model** of the measurand is **not totally correct**

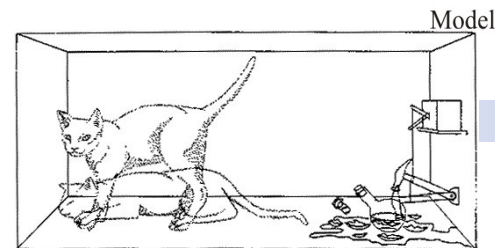


Theory ➔



Real World

➔ **State** of the measurand is **not completely known**



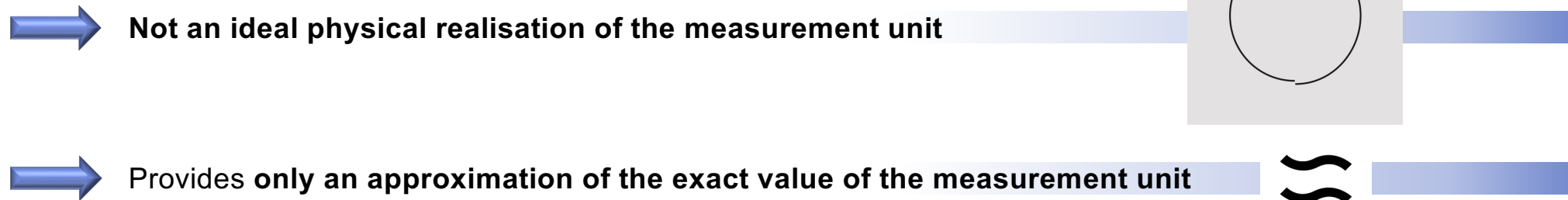
Model

➔ **Measurement process** **modifies** the measurand **state**



## 3.2.2 The measurement process: sources of inaccuracy

### The standard



“All exact science is dominated  
by the idea of approximation.”

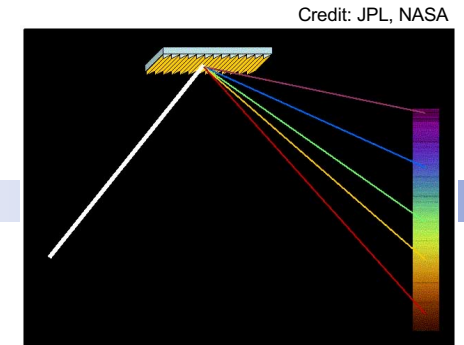
Bertrand Russell

### 3.2.3 The measurement process: sources of inaccuracy

#### The method

➡ The measurement method is **usually based on a physical phenomenon**

➡ It usually **does not take into account other phenomena that may interfere** with the considered one

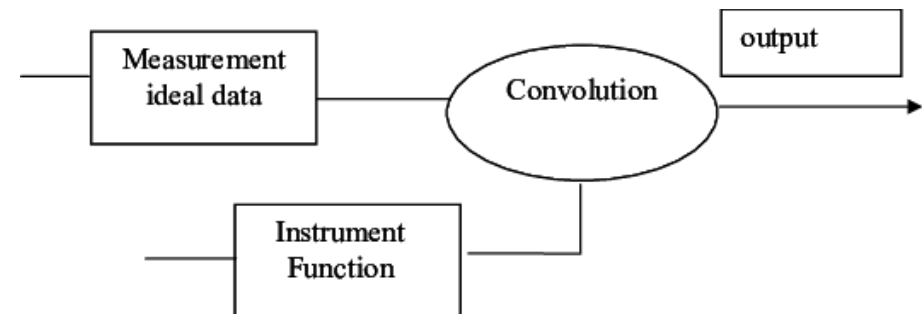


## 3.2.4 The measurement process: sources of inaccuracy

### The instrument

➡ The **actual operating principle** differs from the **ideal one** due to:

- Non-ideal components
- Internally generated noise
- Sensitivity to environmental conditions
- Lack of calibration
- Ageing
- ...



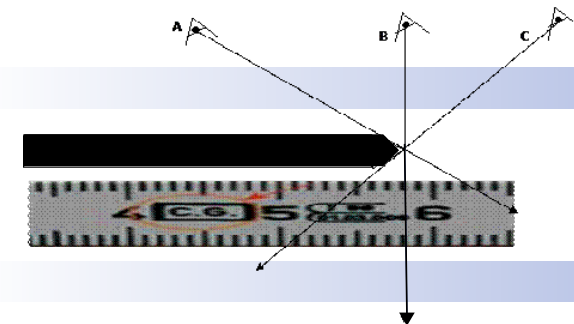
The ideal image is convolved by the instrument function to result in a non-ideal image

Teper, L., Dickstein, P., Ingman, D. (2006). "The line-spread-function of an ultrasonic measurement system: theory and experimental characterization". *Third International Conference on Metrology, IMS*.

## 3.2.5 The measurement process: sources of inaccuracy

### The operator

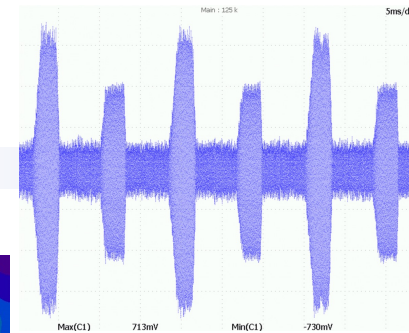
- ➔ **Measurement execution** must be “**just in time**”  
(if measurement conditions are evolving...)
- ➔ **Evaluation of the pointer displacement** on a ruler  
(mind the parallax errors)
- ➔ Correct **evaluation of the measurement results**
- ➔ Correct **post processing of the readings**



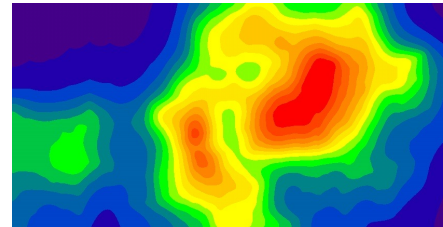
## 3.2.6 The measurement process: sources of inaccuracy

### The rest of the world! (influence quantities)

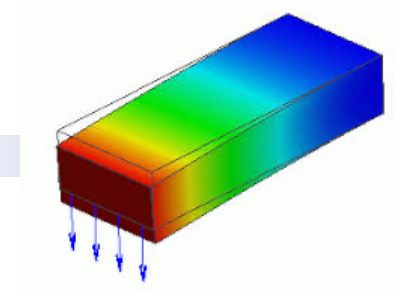
➔ Electromagnetic interferences (EMI)



➔ Temperature variations



➔ Mechanical stress

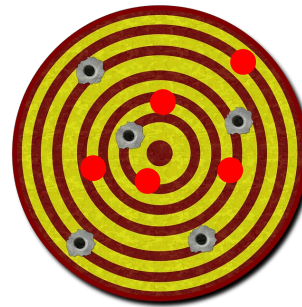


➔ Other effects ... all more or less impacting the final result depending on the kind of measurement!

### 3.3 The measurement process: effects of inaccuracy



If the measurement procedure is **repeated a number of times**, the **obtained measured values always differ from each other**, even if the measurement conditions are not changed



If the measurement procedure is **repeated by another operator, reproducing the same conditions** somewhere else with other instruments (albeit of the same type), **different results are obtained**

### 3.3.1 The measurement process: effects of inaccuracy

#### A first conclusion... and a new problem

➡ **Expressing a measurement result with a single number (and a unit of measurement) is meaningless**

➡ **We cannot use a single measured value in any comparison** with other measurement results obtained by means of the same measurement procedure

➡ **Is this acceptable?**





### 3.3.2 The measurement process: effects of inaccuracy

**Can we still represent the result of a measurement with a single value and avoid the aforementioned drawbacks?**



**Yes, if we find a way to assess how good the measurement result is**

- **How well does the result of a measurement approximate the measurand value?**
- **Need to quantify how complete (or incomplete) the information retrieved from the measurand's measurement result truly is...**



## **4. Uncertainty of measurement (also: error theory versus uncertainty)**



## 4.1 Error vs Uncertainty

### Old way: the error theory

➡ **Axiom: the true value  $x_t$  of measurand  $x$  is somehow known**

➡ **If  $x_m$  is the measured value, an absolute error is defined as:**

$$\varepsilon = x_m - x_t$$

➡ **If the maximum error  $\varepsilon_M$  is considered, an interval is obtained about the measured value:**

$$[x_m - \varepsilon_M ; x_m + \varepsilon_M]$$

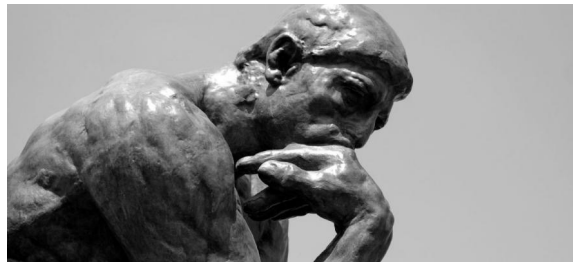
## 4.1 Error vs Uncertainty

### Old way: the error theory

➡ Axiom: the true value  $x_t$  of measurand  $x$  is somehow known

### A philosophical question...

➡ If measurement results can provide only incomplete knowledge about the quantity subject to measurement, **how can we ever know the true value of the measurand?**



## 4.1 Error vs Uncertainty

### Old way: the error theory

➡ Axiom: the true value  $x_t$

#### The “true” value issue

➡ The true value is unknown

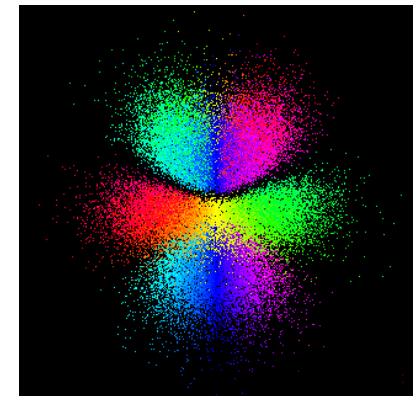
➡ The true value is unknown

➡ In some situations, the true value is unknown

- The measurand can only be approximated



Probability density function





## 4.1 Error vs Uncertainty

### The uncertainty concept



Introduced in the late 1980s



**Uncertainty as a quantifiable attribute** of the measurement, able **to assess the quality** of the measurement process and result



The concept comes from the awareness that:

*“when all known or suspected components of error have been evaluated, and the appropriate corrections have been applied, there still remains an uncertainty about the correctness of the stated results, that is, a doubt about how well the result of the measurement represents the value of the quantity being measured”*

*Uncertainty of measurement, Part 3: Guide to the Expression of Uncertainty in Measurement (GUM:1995), ISO/IEC*



## 4.2 Uncertainty of measurement

### What is uncertainty of measurement?

➡ Uncertainty of measurement is **the doubt that exists about the result of any measurement.**

### Uncertainty meaning

➡ A **parameter**, associated with the result of a measurement, that **characterises the dispersion of the values** that could *reasonably* be **attributed to the measurand**

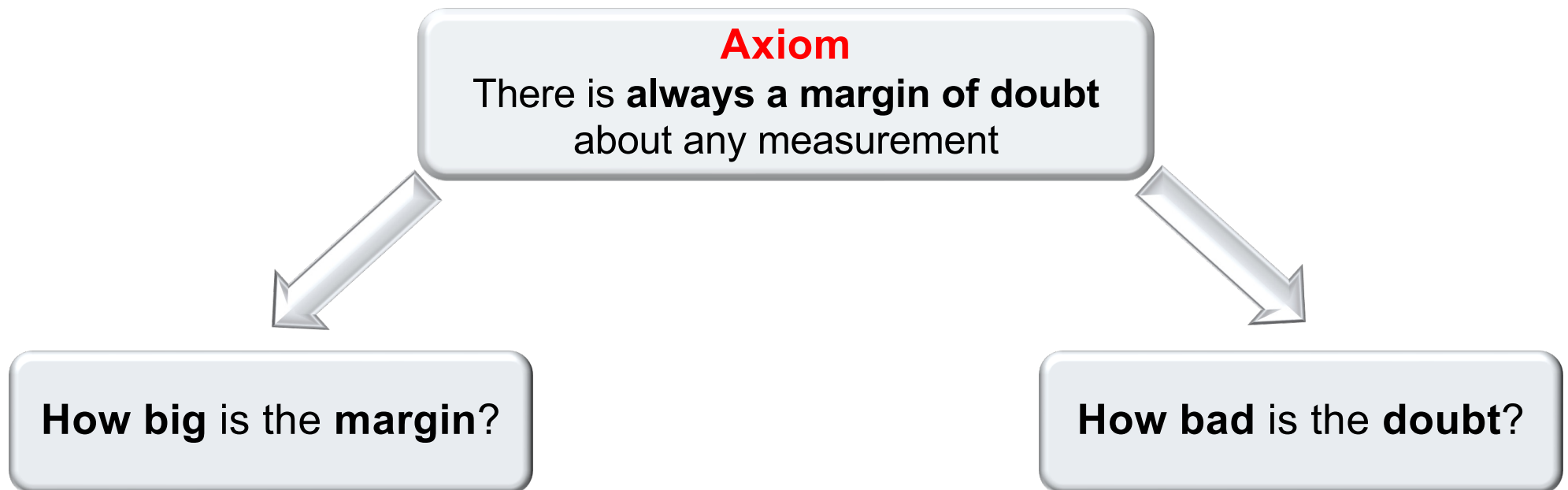
➡ The adverb *reasonably* is the key point in this definition:

- the **capability** of quantifying the dispersion in a reasonable way **depends on** the amount of **available information**



## 4.2 Uncertainty of measurement

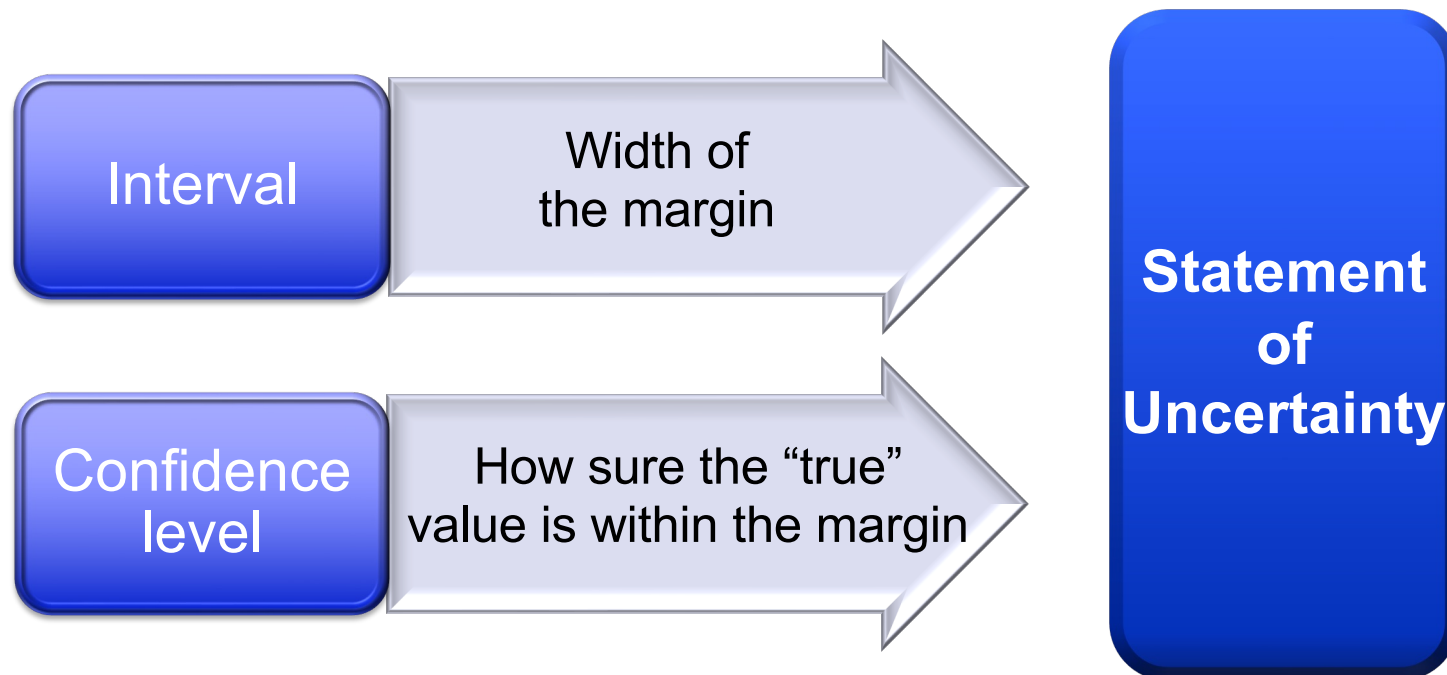
### Expressing uncertainty of measurement





## 4.2 Uncertainty of measurement

### Expressing uncertainty of measurement



## 4.2 Uncertainty of measurement

### Expressing uncertainty of measurement

#### *Example*



**Stick length:**

we find that it is 60 cm plus or minus 2 cm,  
at the 95 percent confidence level

**Result:**

**$60 \text{ cm} \pm 2 \text{ cm}$ , at a level of confidence of 95%.**



**Uncertainty statement**



## 4.3 Uncertainty of measurement: why is it important?

➡ **Calibration** – where the uncertainty of measurement must be reported on the certificate

➡ **Test** – where the uncertainty of measurement is needed to determine a pass or fail

➡ **Tolerance** – where you need to know the uncertainty before you can decide whether the tolerance is met





## 4.4 What is not a measurement uncertainty?

- ➡ **Mistakes made by operators** are not measurement uncertainties – they don't contribute to uncertainty
- ➡ **Tolerances** are not uncertainties: they are acceptance limits chosen for a process or a product
- ➡ **Specifications** are not uncertainties: they tell you what to expect from a product (including non-technical qualities of the item, such as its appearance)
- ➡ **Accuracy (or rather inaccuracy)**: a qualitative term (“a measurement was accurate/inaccurate”) – uncertainty is quantitative (when a  $\pm$  figure is quoted, it may be called an uncertainty – not an accuracy)

## 4.5 'Error' in the age of uncertainty

-  ~~A measured value will differ from the "true" (haha!) value for several reasons~~
-  A measured value will differ from the value of a particular quantity intended to be measured (aka, the measurand) for several reasons, *some of which we may know about*  
I.e.: the quantity realised for measurement is an approximation of the measurand
-  To account for known differences, we apply a **correction**  
E.g.: the measured value may be multiplied by a gain determined during the instrument's calibration
-  This correction will never be perfectly known and there will also be other effects that cannot be corrected, so after correction there will always be a residual, unknown error
-  This specific error is conceptualised as a draw from a probability distribution function, and the *uncertainty* associated with the measured value is *a measure of that probability distribution function*
-  Final (unknown) error on the measured value is drawn from the overall probability distribution described by the uncertainty associated with the measured value – this is **built up from the probability distributions associated with all the different sources of uncertainty**



## 4.5 'Error' in the age of uncertainty

-  ~~The final corrected result is viewed as the best estimate of the “true” value of the measurand~~
-  The final corrected result is simply the best estimate of the value of the quantity intended to be measured (*i.e.*, the approximately defined measurand, itself a source of uncertainty)

### Reference reading:

JCGM 100:2008 – GUM 1995 with minor corrections, *Annex D: “True” value, error, and uncertainty*, 1<sup>st</sup> ed. 2008.



## **5. Basic statistics on sets of numbers: from getting the best estimate to calculating an estimated standard deviation**



## 5.1 Getting the best estimate

### Taking the average of a number of readings

➡ An average gives you an estimate of the value of the measurand

➡ Average (arithmetic mean): 
$$\bar{x} = \frac{1}{n} * \sum_{i=1}^n x_i$$

➡ The more measurement results are used, the better the estimate.



## 5.2 Spread... quantified using standard deviation

- ➡ The **spread of values** tells us something about the uncertainty of a measurement
- ➡ **Standard deviation**: how different the individual readings typically are from the average of the set
- ➡ The “true” value for the the **standard deviation** can only be found from a very large (infinite) set of readings
- ➡ **Estimated standard deviation** formula:  $s = \sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{(n - 1)}}$   
*Also known as **standard uncertainty***

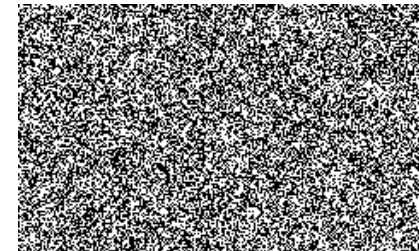


## **6. The general kinds of uncertainty in any measurement: random or systematic, and distribution (the ‘shape’ of the errors)**

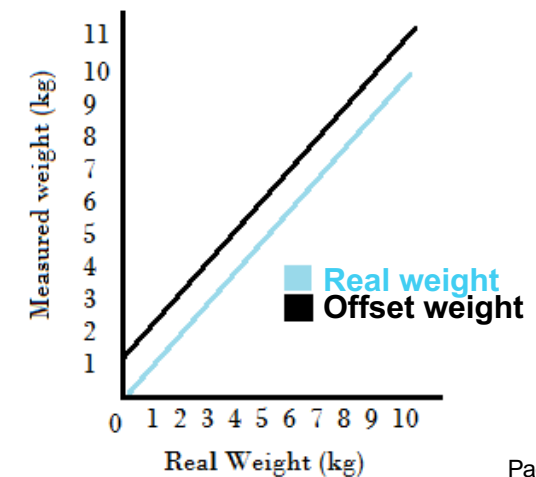
## 6.1 Random or systematic

The **effects** that give rise to uncertainty in measurement can be either:

- **random** – where repeating the measurement gives a randomly different result  
  
**the more measurements are averaged, the better the estimate**
- **systematic** – where the same influence affects the result for each of the repeated measurements



A randomly generated bitmap, made using RANDOM.ORG's True Random Number Generator (TRNG)



## 6.2 Distribution – the ‘shape’ of the errors

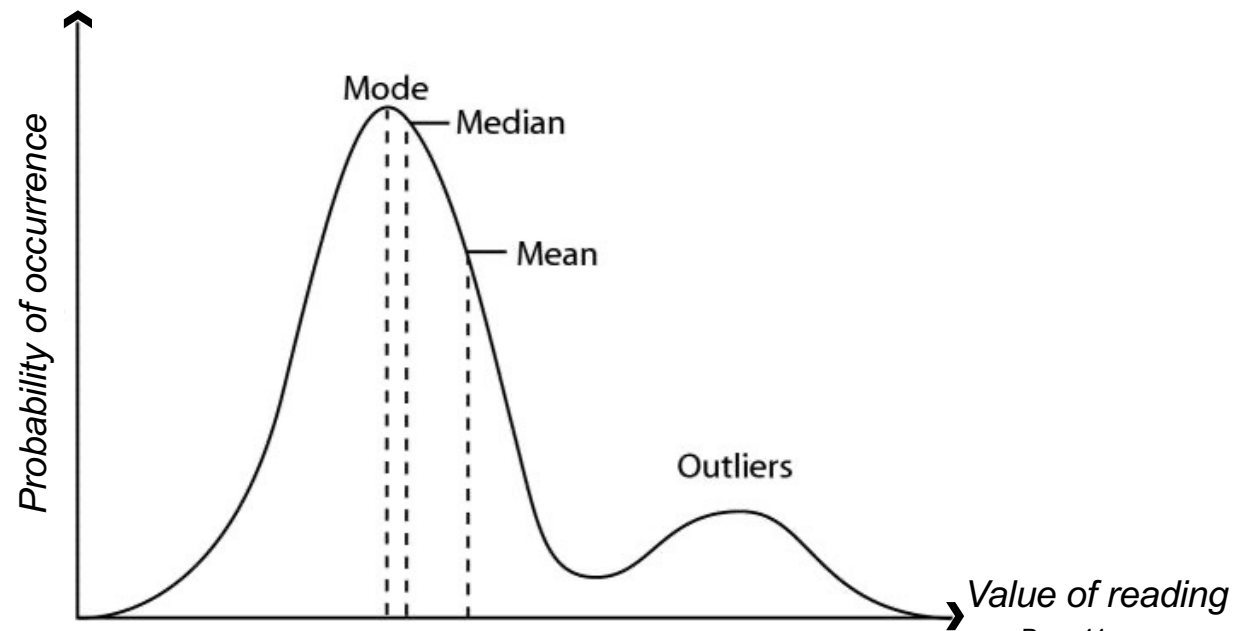
The spread of a set of values can take different forms

➡ ***probability distributions***

The different elements of a probability distribution:

- mode: the number that appears most frequently
- median: middle point between higher and lower half of the distribution
- mean: average
- outliers: unusual/extreme values, have the effect of skewing the data

Basics of Uncertainty of Measurement, S. Trim



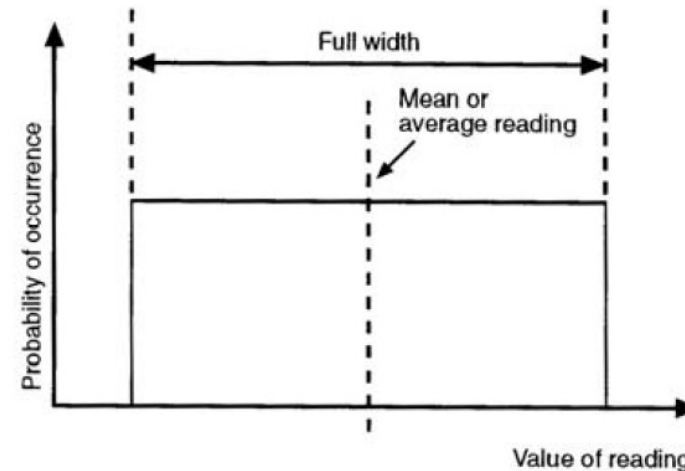
## 6.2 Distribution – the ‘shape’ of the errors

### The uniform or rectangular distribution

When the measurements are quite evenly spread between the highest and the lowest values, a rectangular or uniform distribution is produced. Example: rain drops falling on a thin, straight telephone wire, would be as likely to fall on any one part as on another.

Probability density function (in terms of the mean  $\mu$  and variance  $\sigma^2$ ):

$$f(x) = \begin{cases} \frac{1}{2\sigma\sqrt{3}} & \text{for } -\sigma\sqrt{3} \leq x - \mu \leq \sigma\sqrt{3} \\ 0 & \text{otherwise} \end{cases}$$

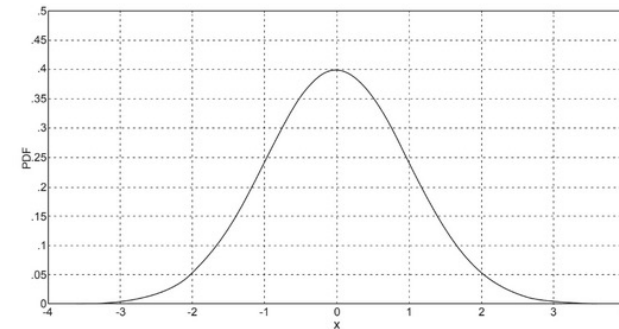


## 6.2 Distribution – the ‘shape’ of the errors

### The normal (*aka* Gaussian) distribution

In a set of readings, sometimes the **values are more likely to fall near the average than further away** – typical of a *normal* or *Gaussian* distribution:

- physical quantities that are expected to be the sum of many independent processes (such as measurement errors) often have distributions that are approximately normal
- Standard formulation:  $N(\mu, \sigma^2)$  where
  - $\mu$  is the mean: controls the location of the peak of the distribution
  - $\sigma^2$  is the variance: a larger value for  $\sigma^2$  results in a ‘fatter’ “bell curve” (distribution shape)
- Probability density function (PDF) is:  $p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
- *Note that:*
  - 68.258% of the data is under the curve between  $\mu \pm 1\sigma$
  - 95.45% of the data is under the curve between  $\mu \pm 2\sigma$
  - 99.73% of the data is under the curve between  $\mu \pm 3\sigma$





## 6.2 Distribution – the ‘shape’ of the errors

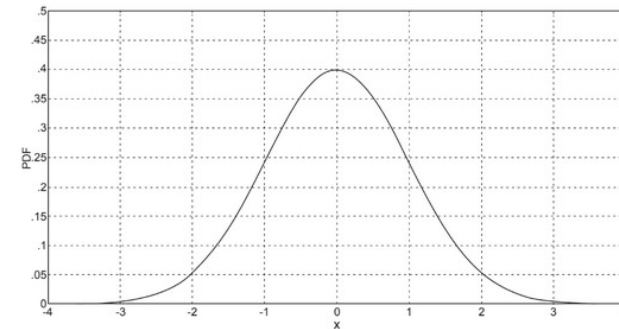
### The normal (*aka* Gaussian) distribution

Out of all distributions with mean  $\mu$ , variance  $\sigma^2$ , and support over all of  $\mathbb{R}$ , the normal distribution,  $N(\mu, \sigma^2)$ , **maximises entropy**.

The class of normal distributions  $N(\cdot, \sigma^2)$  all maximize entropy subject to the constraints of a fixed variance  $\sigma^2$  and support over  $\mathbb{R}$ .

*About the notion of entropy:–*

The **entropy of a distribution is a measure of how flat and smooth that distribution is**; the flatter and smoother a distribution, the higher its entropy. The flattest and smoothest distribution over  $\mathbb{R}$  is the uniform distribution, and the flattest and smoothest distribution over  $\mathbb{R}$  with a given finite mean and variance is the normal distribution with that mean and variance.



## 6.2 Distribution – the ‘shape’ of the errors

### The log-normal distribution

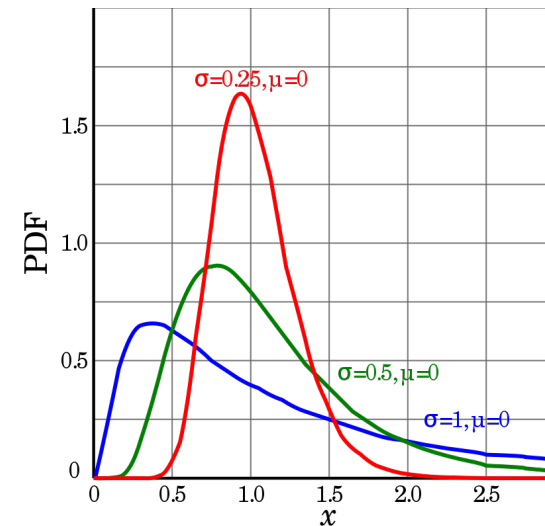
A log-normal distribution is a continuous probability distribution of a random variable whose logarithm is normally distributed.

⇒ If random variable  $X$  is log-normally distributed, then  $Y = \ln(X)$  has a normal distribution.

⇒ If  $Y$  has a normal distribution, then the exponential function of  $Y$ ,  $X = \exp(Y)$ , has a log-normal distribution.

A random variable which is log-normally distributed takes only positive real values.

A log-normal process is the statistical realisation of the **multiplicative product** of many independent random variables, **each of which is positive**. This is justified by considering the central limit theorem in the log domain. The log-normal distribution is the maximum entropy probability distribution for a random variate  $X$  for which the mean and variance of  $\ln(X)$  are specified.





## **7. How to calculate uncertainty of measurement: the 2 ways to estimate uncertainties, and the 8 main steps to evaluating uncertainty**

## 7.1 The two ways to estimate uncertainties

Firstly: –

Identify sources of  
uncertainty in the  
measurement



Estimate the size of  
the uncertainty from  
each source



Combine the  
individual  
uncertainties to give  
an overall figure

### Type A evaluations

Uncertainty estimates using statistics  
(usually from repeated readings)

### Type B evaluations

Uncertainty estimates from any other  
information

There is a temptation to think of '**Type A**' as '**random**' and '**Type B**' as '**systematic**', but this is not necessarily true.

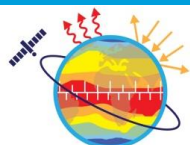


# The steps to an uncertainty budget

Emma Woolliams

4 April 2017

Prepared by Paul Miller



Metrology for Earth  
Observation and Climate

<http://www.emceoc.org>

**EMRP**

European Metrology Research Programme  
■ Programme of EURAMET

The EMRP is jointly funded by the EMRP participating countries  
within EURAMET and the European Union



# At the end of this module, you should understand

- Uncertainty analysis is a multi-step process

- Understanding the problem

- Determining the formal relationships

- Propagating the uncertainties

- How to develop an uncertainty budget

- 8 steps to an uncertainty budget

- There is no single right way

- Mathematical / modelling
  - Experimental
  - Combination

# 8 steps to an uncertainty budget

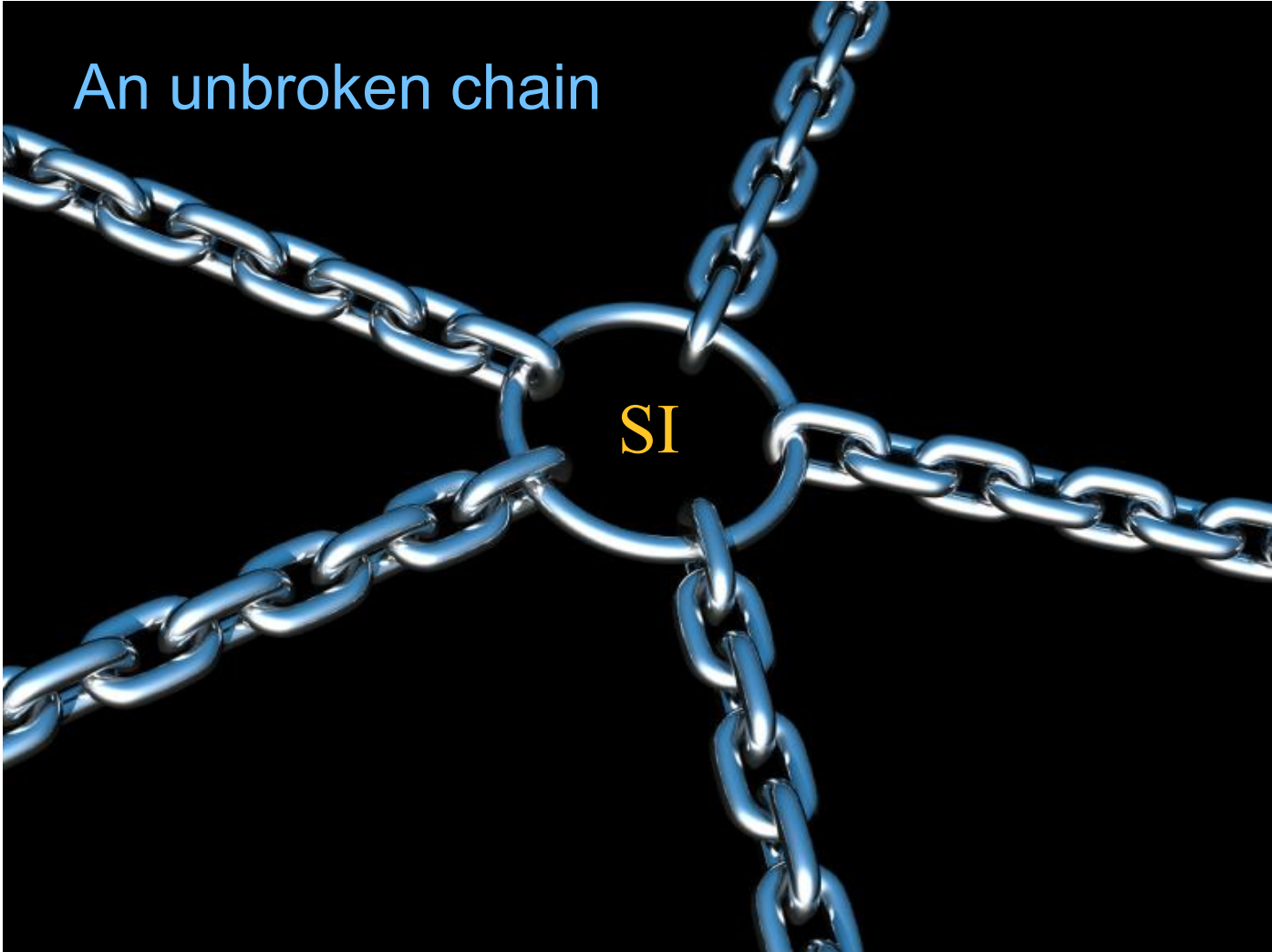
- Understanding the problem
  - Step 1: Describing the Traceability Chain
  - Step 2: Writing down the calculation equations
  - Step 3: Considering the sources of uncertainty
- Determining the formal relationships
  - Step 4: Creating the measurement equation
  - Step 5: Determining the sensitivity coefficients
  - Step 6: Assigning uncertainties
- Propagating the uncertainties
  - Step 7: Combining and propagating uncertainties
  - Step 8: Expanding uncertainties

# Understanding the problem

- Understanding the problem
  - Step 1: Describing the Traceability Chain**
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An unbroken chain

SI

The image shows four interlocking metal chains, likely made of steel, arranged in a cross pattern. The chains are composed of oval-shaped links. In the center, where the four chains intersect, they form a circular void. Inside this central circle, the letters "SI" are printed in a yellow, serif font. The background is a solid black, which makes the metallic chains stand out. The lighting is dramatic, highlighting the reflective surfaces of the metal links.

# Describing the Traceability Chain

SI Units



**Cryogenic  
radiometer**



**Primary  
Standard**



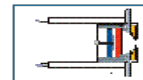
**Laser**



**Reference  
photodiode**



**Laser**



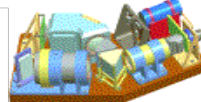
**Filter-  
radiometer**

**Radiance  
(T via Planck)**



**Blackbody  
3500 K**

**Spectrometer  
Radiance /  
Irradiance**



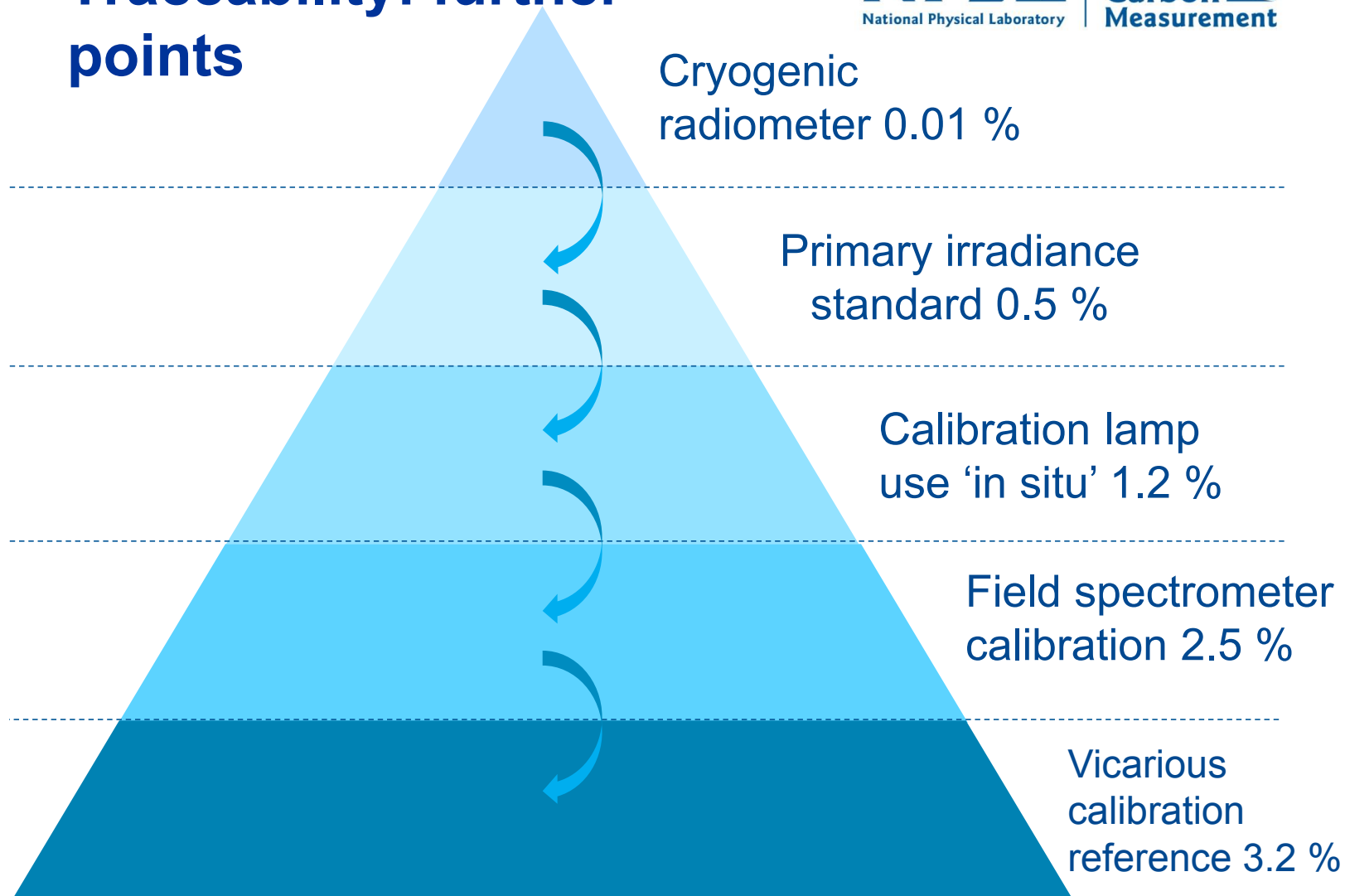
**Satellite  
Earth Imager**



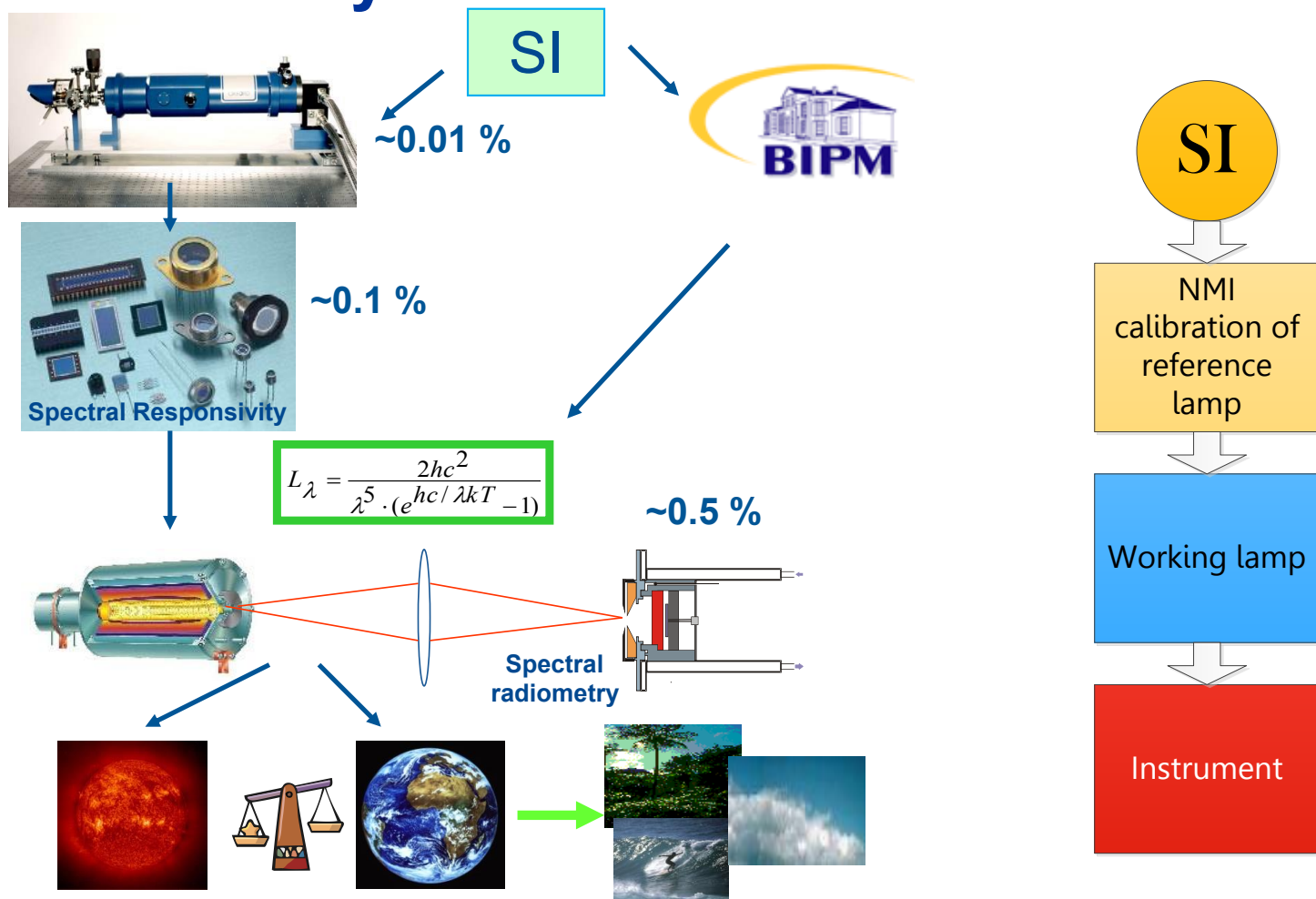
**Standard  
lamp**



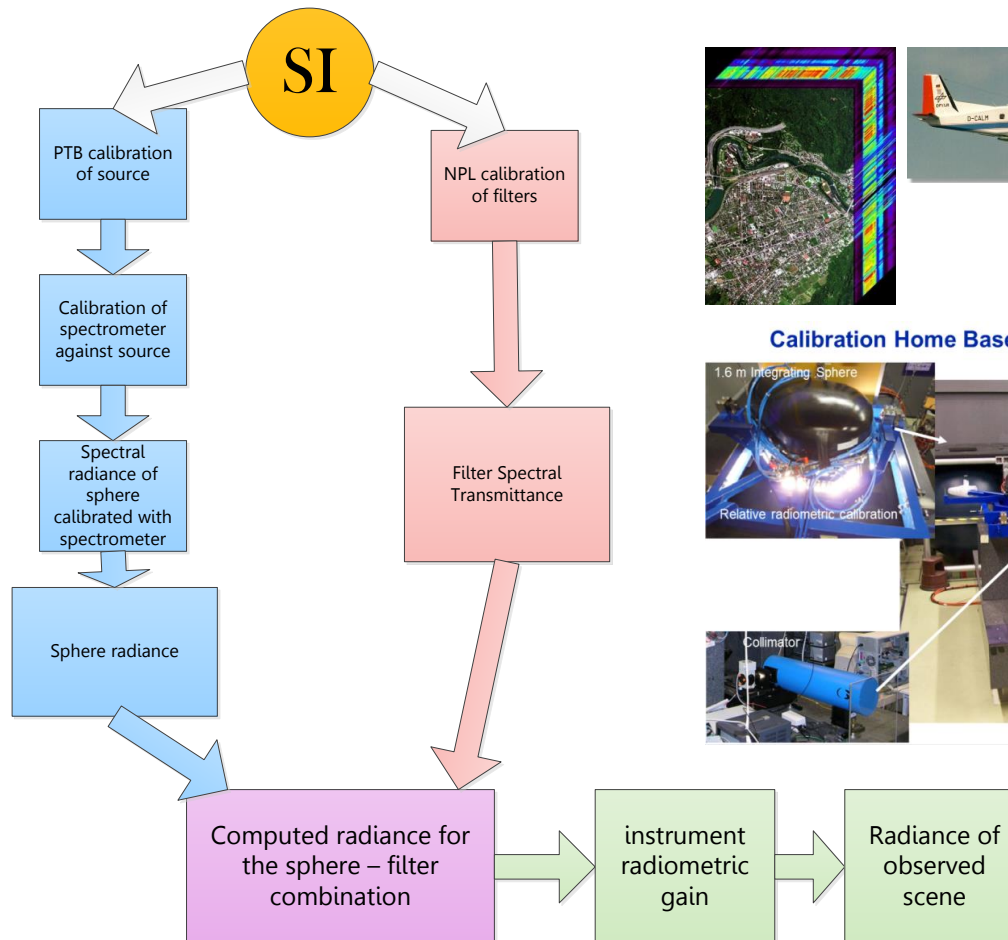
## Traceability: further points



# Describing the Traceability Chain



# Describing the Traceability Chain



# Understanding the problem

- Understanding the problem
  - Step 1: Describing the Traceability Chain
  - Step 2: Writing down the calculation equations**
  - Step 3: Considering the sources of uncertainty
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  - Step 7: Combining and propagating uncertainties
  - Step 8: Expanding uncertainties

# Writing down the calculation equations



$$L_s = \frac{E_{\text{FEL}} \beta_{0-45}}{\pi}$$

# Understanding the problem

- Understanding the problem
  - Step 1: Describing the Traceability Chain
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- Determining the formal relationships
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# Considering the sources of uncertainty



## Lamp additional effects

- Ageing
- Alignment
- Current stability



## Distance accuracy



## Diffuser additional effects

- Ageing
- Uniformity



## Random noise



# Considering the sources of uncertainty



# Another example of an uncertainty: Reflectance Factor estimation



5 x REF

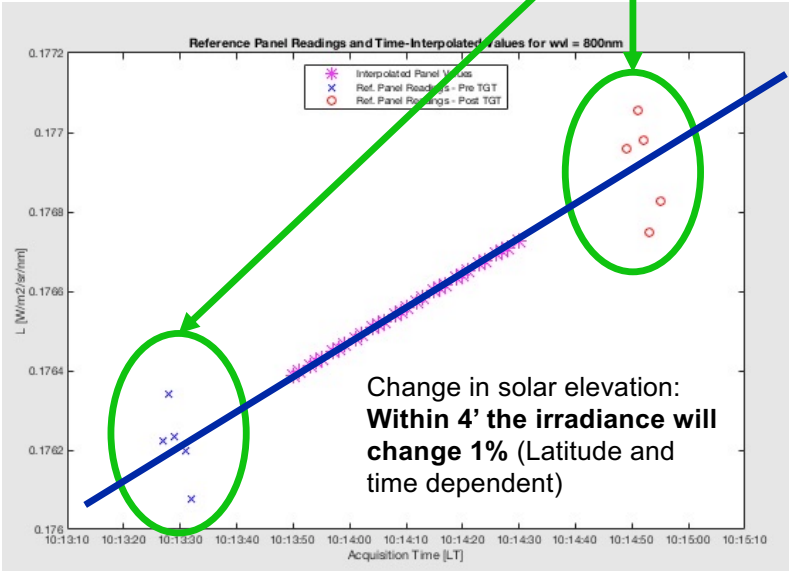
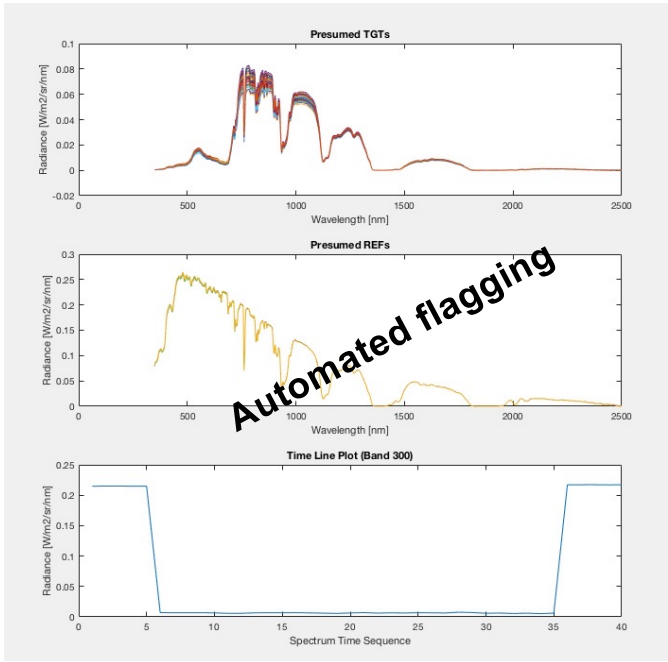
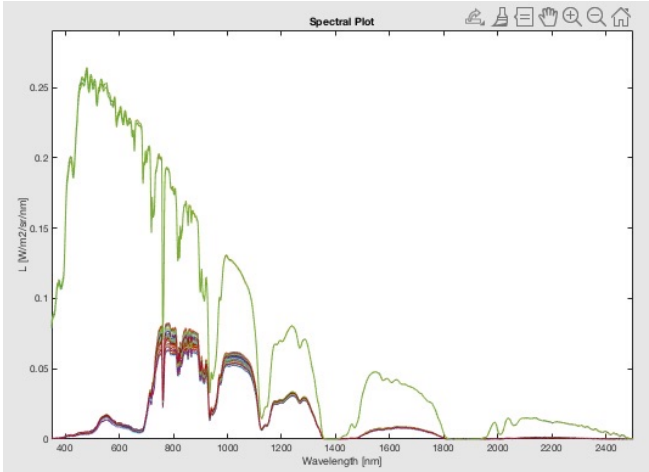


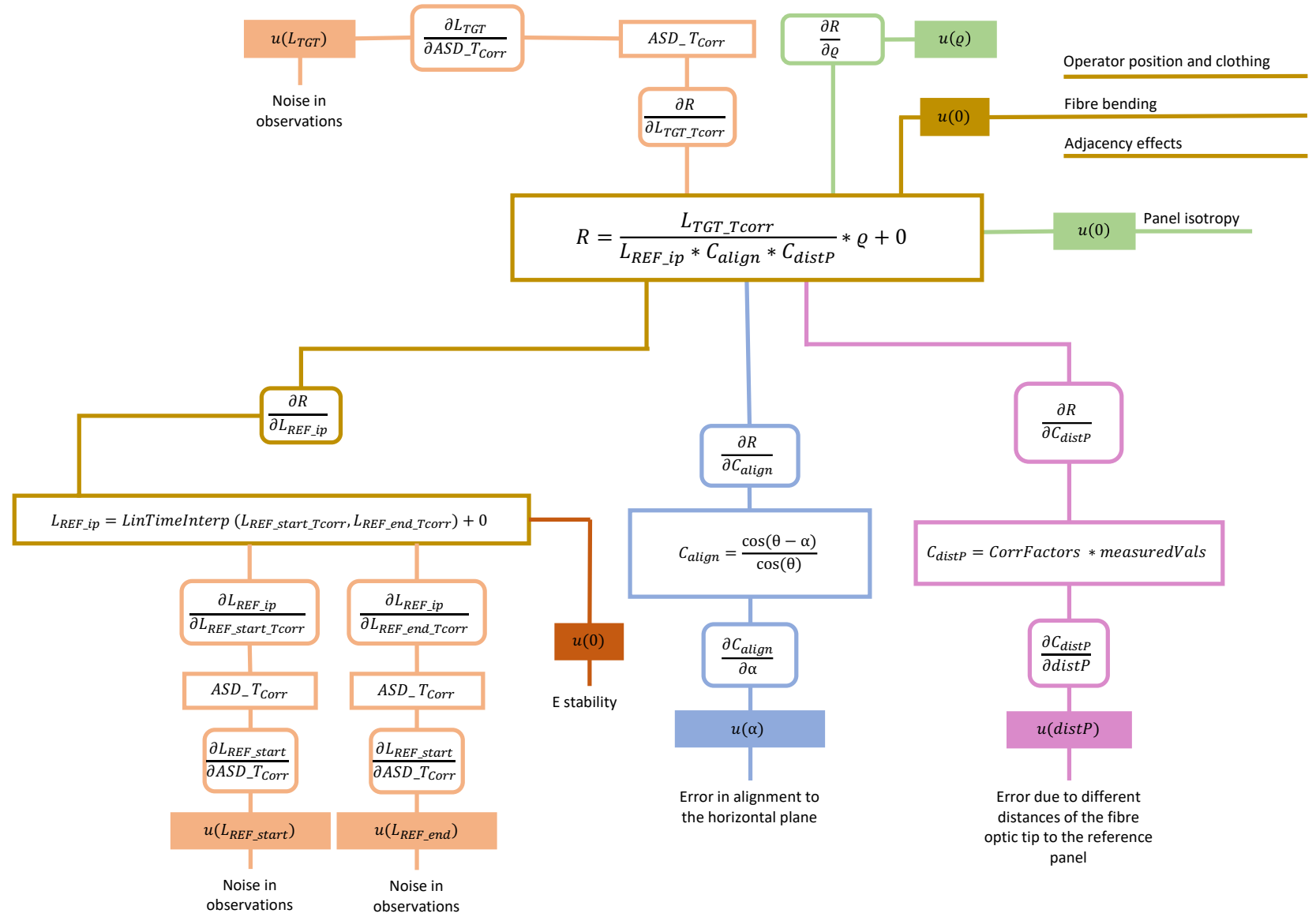
30 x TGT



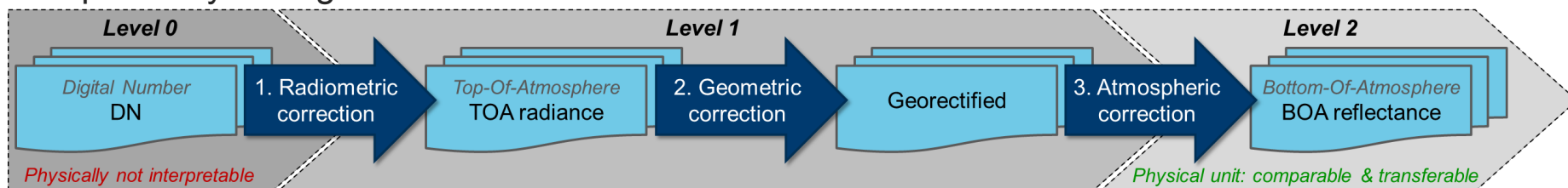
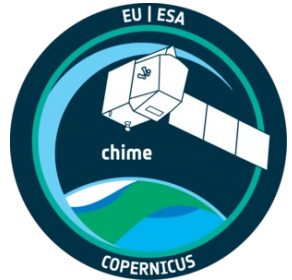
5 x REF

Assumed measurement noise

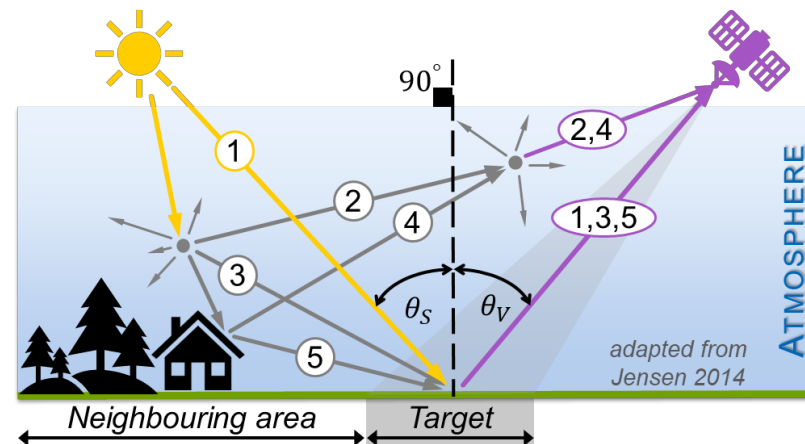




## Another example of an uncertainty: Atmospheric Correction

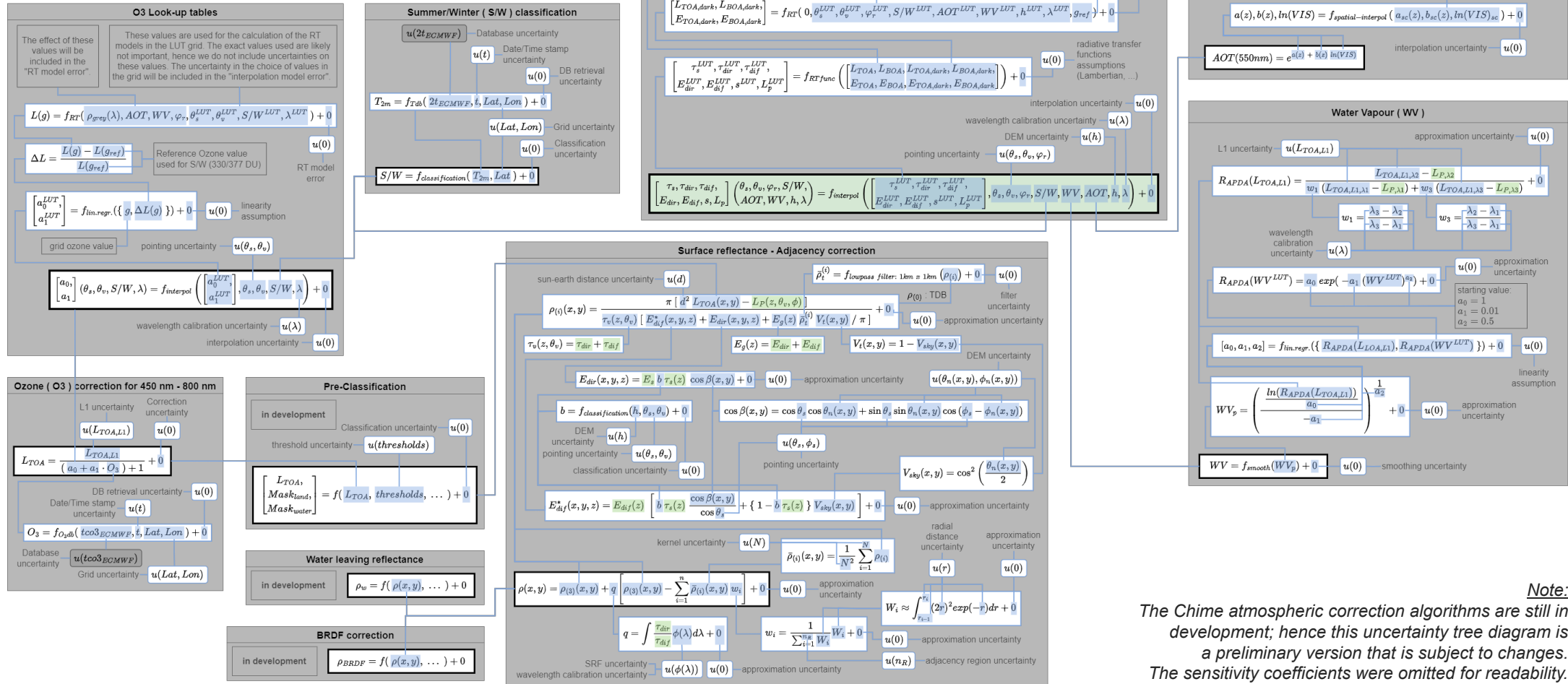


**Atmospheric correction** adjusts for the interaction of electromagnetic energy with the atmosphere using **Radiative Transfer** (RT) models. RT models trace light beams from its source (sun) to a detector (satellite), simulating its absorption, reflection/scattering and transmission. The image shows the solar irradiance ( $E$  yellow) and recorded radiance ( $L$  purple) with its possible pathways in the atmosphere.



# UNCERTAINTY TREE DIAGRAM: CHIME L2A

An uncertainty tree diagram shows the measurement model(s) (*black or blue framed box*) and the origin of each term (*blue, green filled boxes*) as well as their uncertainties ( $u(\dots)$ ). It helps identifying uncertainties and their sources, and shows how they propagate through the system.



**Note:**  
The Chime atmospheric correction algorithms are still in development; hence this uncertainty tree diagram is a preliminary version that is subject to changes. The sensitivity coefficients were omitted for readability.

# Determining the formal relationships

- Understanding the problem
  - Step 1: Describing the Traceability Chain
  - Step 2: Writing down the calculation equations
  - Step 3: Considering the sources of uncertainty
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  - Step 5: Determining the sensitivity coefficients
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  - Step 8: Expanding uncertainties

# Creating the measurement equation



$$L_s = \frac{E_{FEL} \beta_{0-45}}{\pi}$$

# Determining the formal relationships

- Understanding the problem
  - Step 1: Describing the Traceability Chain
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# Determining the sensitivity coefficients

$$u_c^2(y) = \sum_{i=1}^n \left( \frac{\partial f}{\partial x_i} \right)^2 u^2(x_i) + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^n \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} u(x_i, x_j)$$

- There is no single right way
  - **Mathematical / modelling**
  - **Experimental**
  - **Combination**

# Determining the sensitivity coefficients

- Do an experiment



- Analytical expression

$$\begin{aligned} y'_n &= u^2 + 3\sqrt{u} - 1 \quad u = x^2 + 1 \quad y'_x = \dots \\ &= (u^2 + 3\sqrt{u} - 1)_u (x^2 + 1)'_x = (2u + \frac{3}{2\sqrt{u}}) \cdot 2x \\ &= (2(x^2 + 1) + \frac{3}{2\sqrt{x^2 + 1}}) \cdot 2x \\ &= (2x^2 + 2 + \frac{3}{\sqrt{x^2 + 1}}) \cdot 2x \\ &= 4x^3 + 4x + \frac{6x}{\sqrt{x^2 + 1}} \\ &= \lim_{x \rightarrow a} \sqrt[p]{f(x)} = \sqrt[p]{\lim_{x \rightarrow a} f(x)} \\ &A \lim_{x \rightarrow a} b^{f(x)} = b^A \quad b = \text{const}, \lim_{x \rightarrow a} f(x) = A \\ &\lim_{x \rightarrow a} \log_c f(x) = \log_c [\lim_{x \rightarrow a} f(x)], \quad c = \text{const} \end{aligned}$$

- Model it





## Partial Derivatives? Wolfram Alpha to the rescue!

<https://www.wolframalpha.com/input/?i=a%2Fx>

FROM THE MAKERS OF WOLFRAM LANGUAGE AND MATHEMATICA



a/x

Derivative

$$\frac{\partial}{\partial x} \left( \frac{a}{x} \right) = -\frac{a}{x^2}$$

# Determining the formal relationships

- Understanding the problem
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# Assigning uncertainties

| Uncertainty component                                           | Associated uncertainty |          | (relative)              | Uncertainty associated with radiance due to this |
|-----------------------------------------------------------------|------------------------|----------|-------------------------|--------------------------------------------------|
|                                                                 | absolute               | relative | Sensitivity coefficient |                                                  |
| Lamp irradiance (calibration)                                   |                        | 0.30%    | 1                       | 0.30%                                            |
| Diffuser reflectance factor (calibration)                       |                        | 0.30%    | 1                       | 0.30%                                            |
| Lamp-diffuser distance (same as calibration distance for lamp)? | 1 mm in 500 mm         | 0.20%    | 2                       | 0.40%                                            |
| Stability of lamp (short term)                                  |                        | 0.10%    | 1                       | 0.10%                                            |
| Stability of lamp (drift/ageing)                                |                        | 0.10%    | 1                       | 0.10%                                            |
| Alignment of lamp                                               |                        |          |                         | 0.05%                                            |
| Current stability of lamp (at 350 nm)                           | 3 mA                   |          |                         | 0.29%                                            |
| Diffuser stability (ageing)                                     |                        | 0.10%    | 1                       | 0.10%                                            |
| Uniformity of diffuser                                          |                        | 0.50%    | 1                       | 0.50%                                            |

$$L_s = \frac{E_{\text{FEL}} \beta_{0-45}}{\pi} \frac{d_{\text{cal}}^2}{d_{\text{use}}^2} K_{\text{lamp\_stab}} K_{\text{align}} K_{\text{current}} K_{\text{diff\_stab}} K_{\text{unif}}$$

# Propagating the uncertainties

- Understanding the problem
  - Step 1: Describing the Traceability Chain
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- Propagating the uncertainties
  - Step 7: Combining and propagating uncertainties**
  - Step 8: Expanding uncertainties

# Combining and propagating uncertainties

$$u_c^2(y) = \sum_{i=1}^n \left( \frac{\partial f}{\partial x_i} \right)^2 u^2(x_i) + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^n \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} u(x_i, x_j)$$

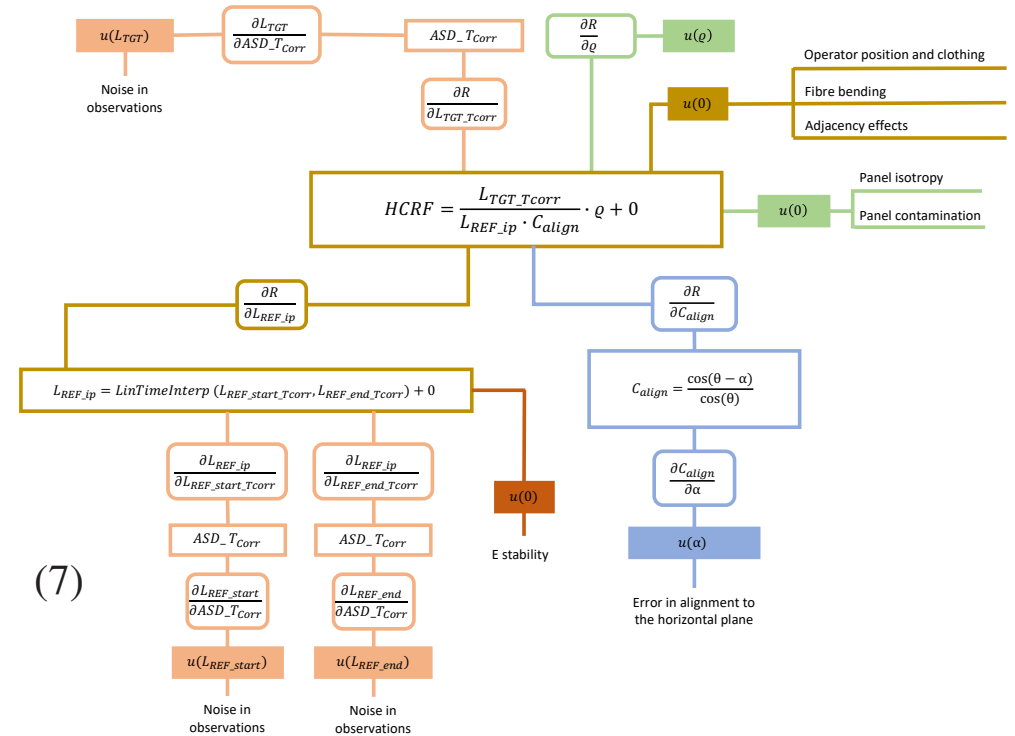
Has a sensitivity coefficient  
Adding in quadrature (% or units)

Averages reduce by  $1/\sqrt{n}$

This term is to do with correlation

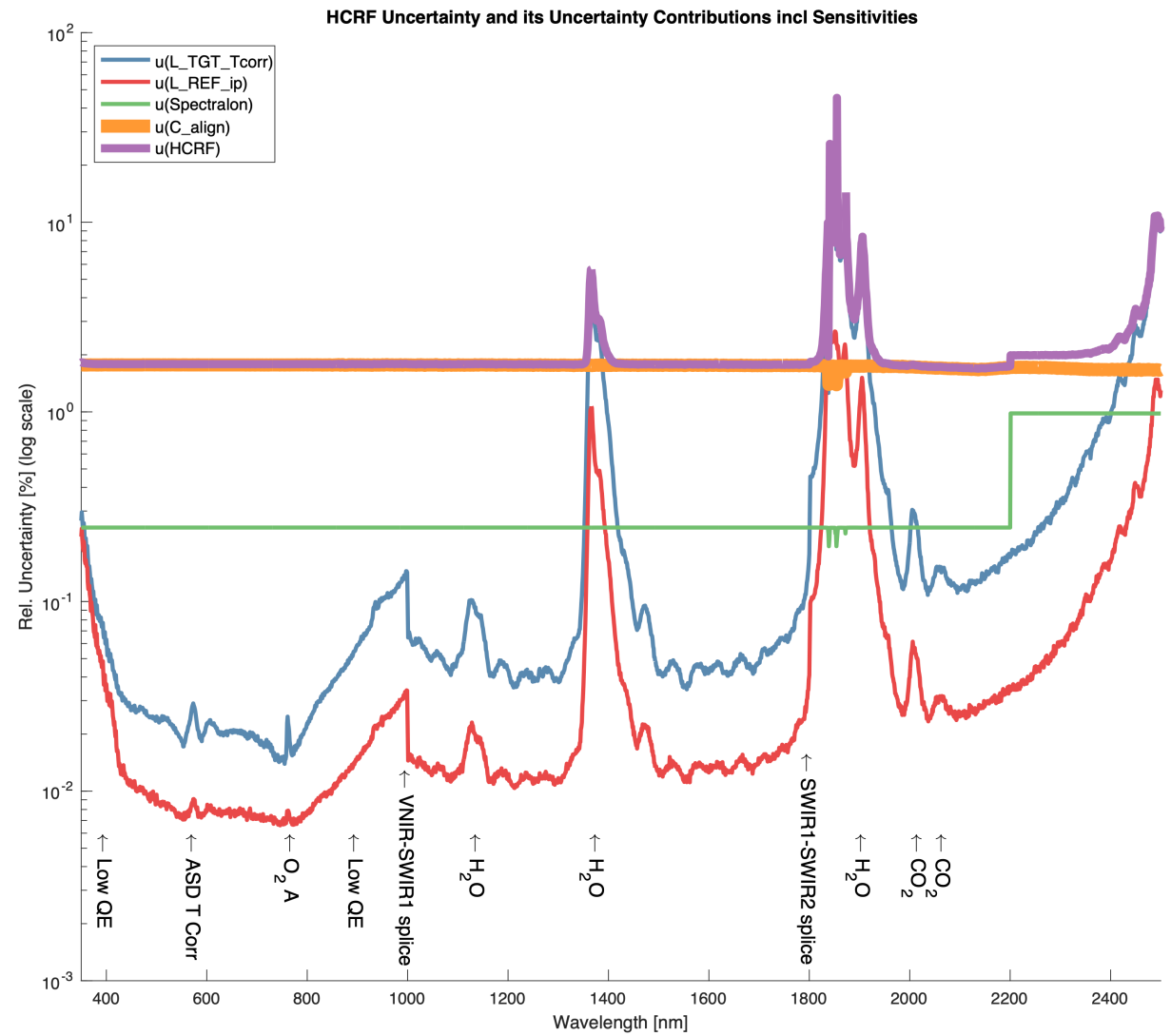
## Analytical propagation for the central equation

$$\begin{aligned}
 u(R) = & \left( \left( \frac{\varrho}{L_{REF_{ip}} \cdot C_{align}} \right)^2 \cdot u(L_{TGT\_Tcorr})^2 \right. \\
 & + \left( -\frac{L_{TGT\_Tcorr} \cdot \varrho}{L_{REF_{ip}}^2 \cdot C_{align}} \right)^2 \cdot u(L_{REF_{ip}})^2 \\
 & + \left( \frac{L_{TGT\_Tcorr}}{L_{REF_{ip}} \cdot C_{align}} \right)^2 \cdot u(\varrho)^2 \\
 & \left. + \left( -\frac{L_{TGT\_Tcorr} \cdot \varrho}{L_{REF_{ip}} \cdot C_{align}^2} \right)^2 \cdot u(C_{align})^2 \right)^{1/2} \cdot (7)
 \end{aligned}$$





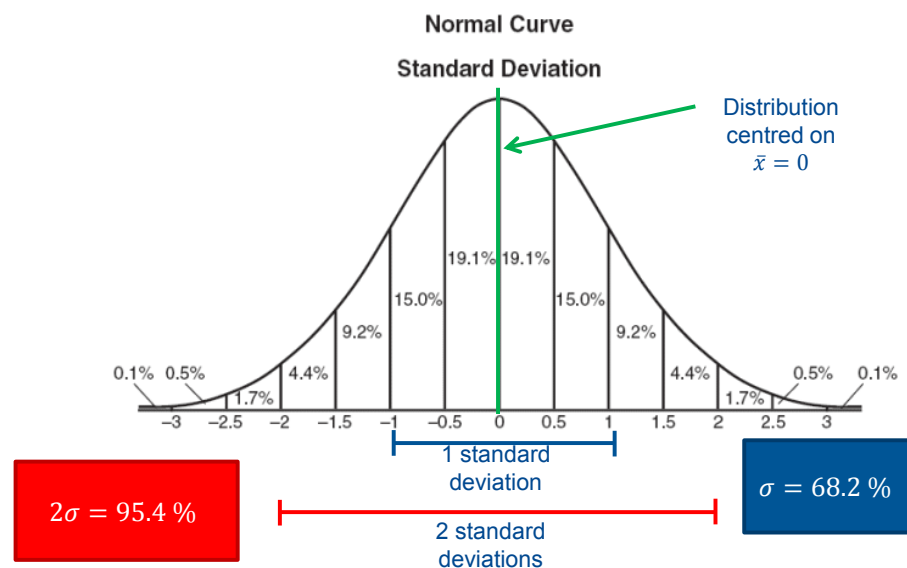
## Uncertainty components and combined uncertainty



# Propagating the uncertainties

- Understanding the problem
  - Step 1: Describing the Traceability Chain
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# Expanding uncertainties



If the distribution is not Gaussian, then a different coverage factor is needed.



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# Introduction to Monte Carlo and the Monte Carlo propagation of uncertainties

ahueni



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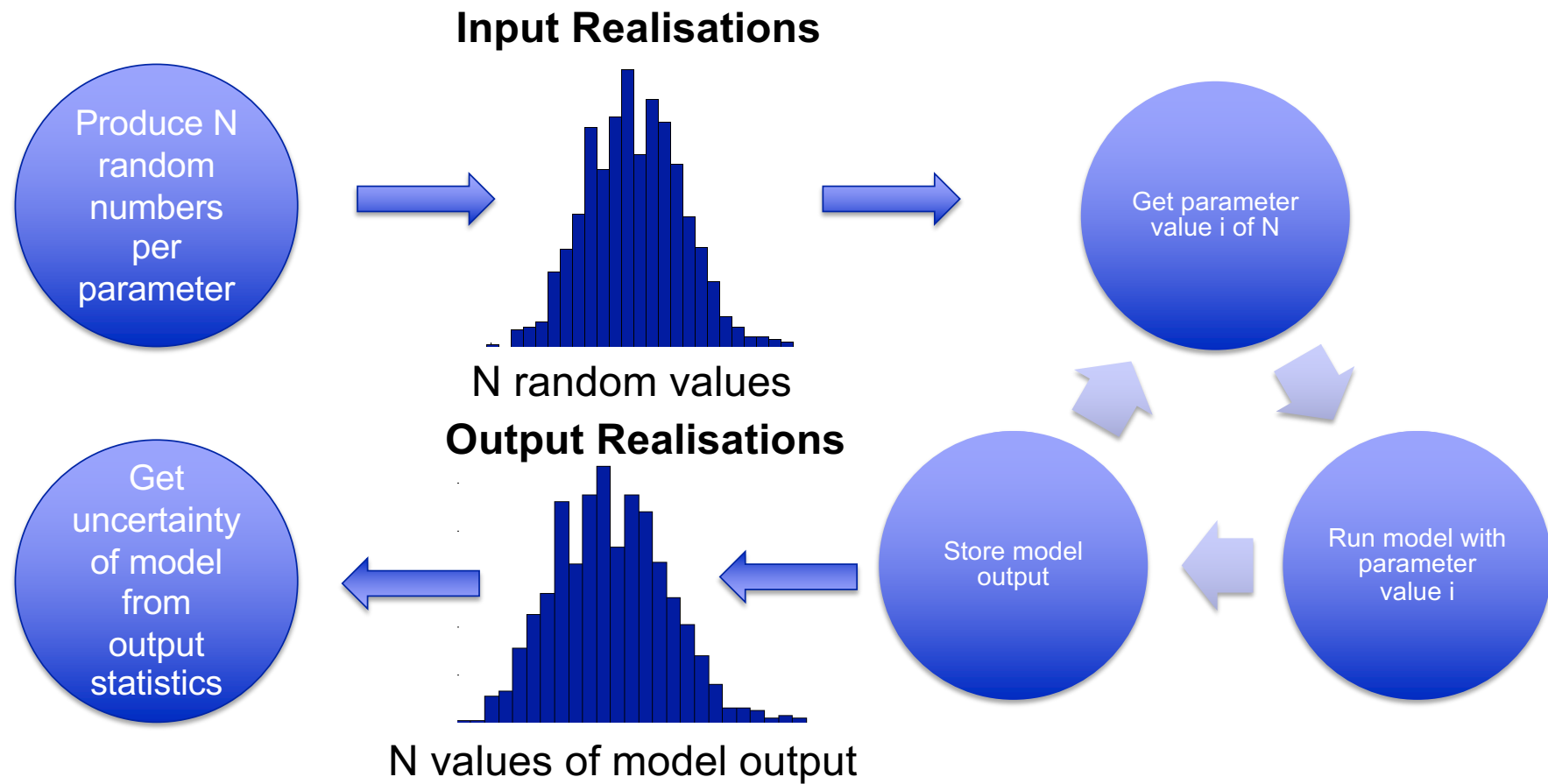


## **Agenda**

- Introduction to MC
- Example of a radiometric calibration
- Application of CoMet



## How Monte Carlo works: Computational Brute-Forcing





Stddev = 1, Rel. Stddev = 2.3810 %

```
% creates realisations, either using a new random generated number  
array or  
% using the supplied random number array  
function [realisations, random_numbers] =  
get_realisations_gauss_dist(numSamples, mu, sigma, random_numbers)  
  
    if nargin == 3  
        random_numbers = randn(numSamples, length(mu));  
    end  
  
    for i=1:numSamples  
        realisations(:,i) = mu + sigma.*random_numbers(i,:);  
    end  
  
end
```

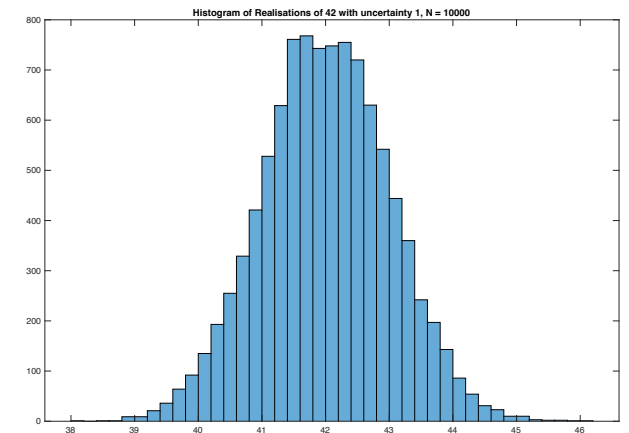
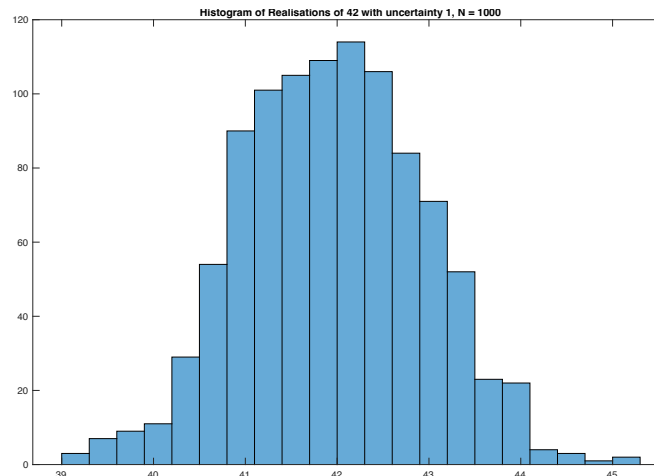
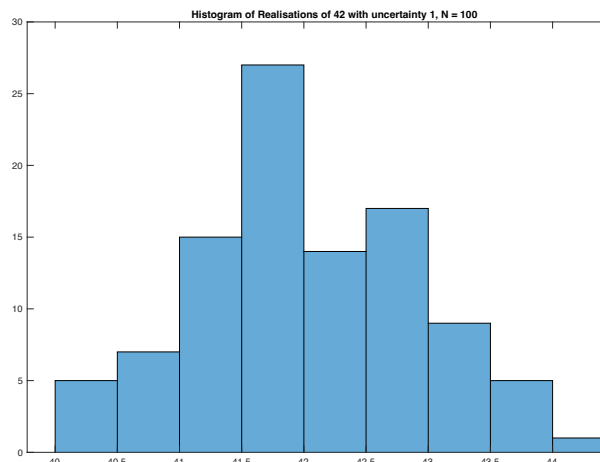
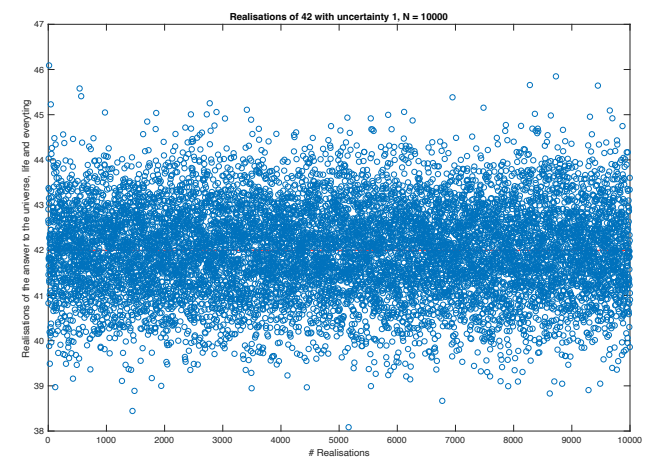
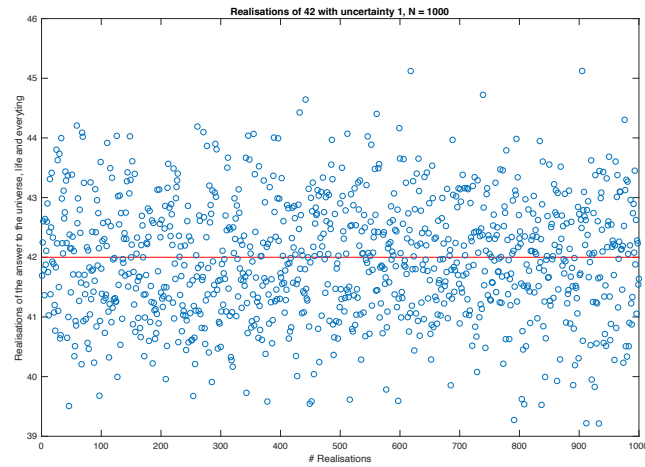
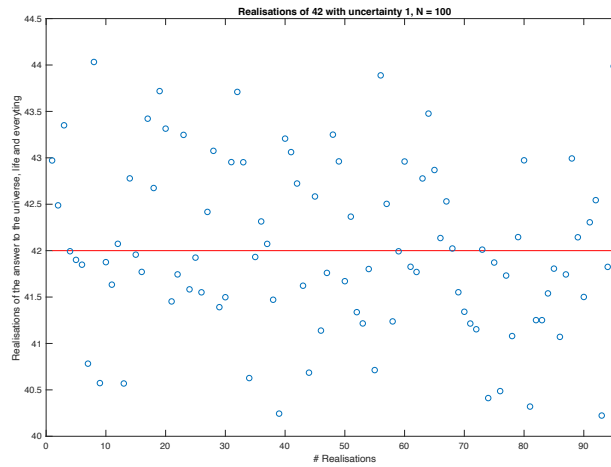


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# Testing the Realisation Generation: Is it Gaussian?

RSL  
measurements | products | policy





## Uncertainty Propagation: Creating the Output Realisations

A squared log transformation of the answer to the universe, life and everything

...

```
% simple model to draw from the generated realisations

output = zeros(size(realisations)); % allocate output vector

% loop over all realisations and calculate the model output
for i=1:length(realisations)

    parameter = realisations(i);

    output(i) = (log(parameter))^2;

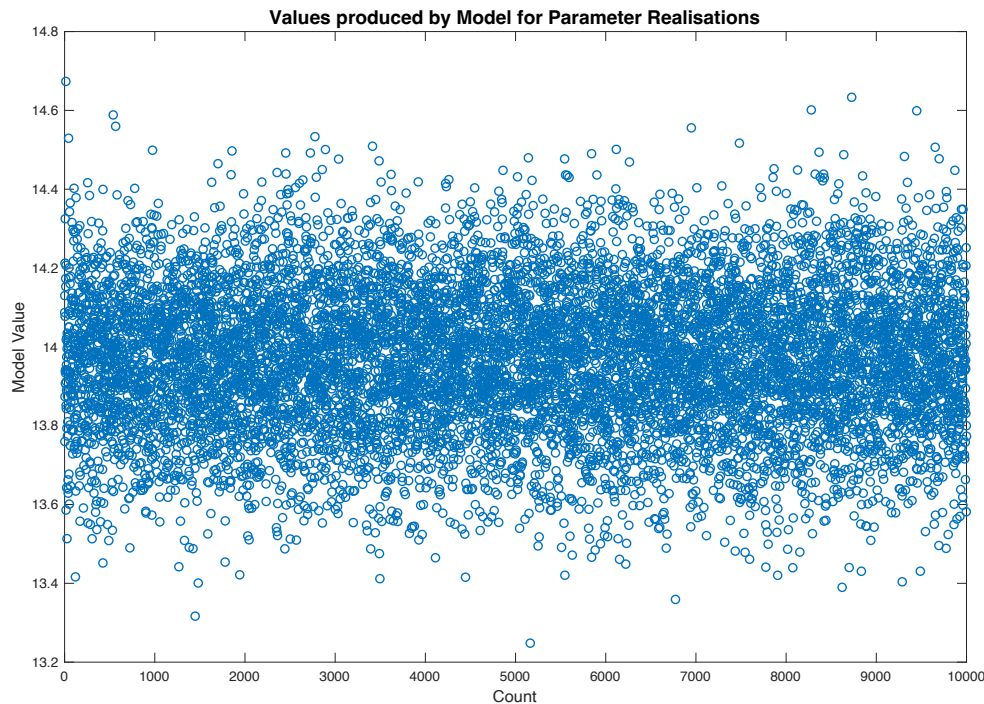
end
```



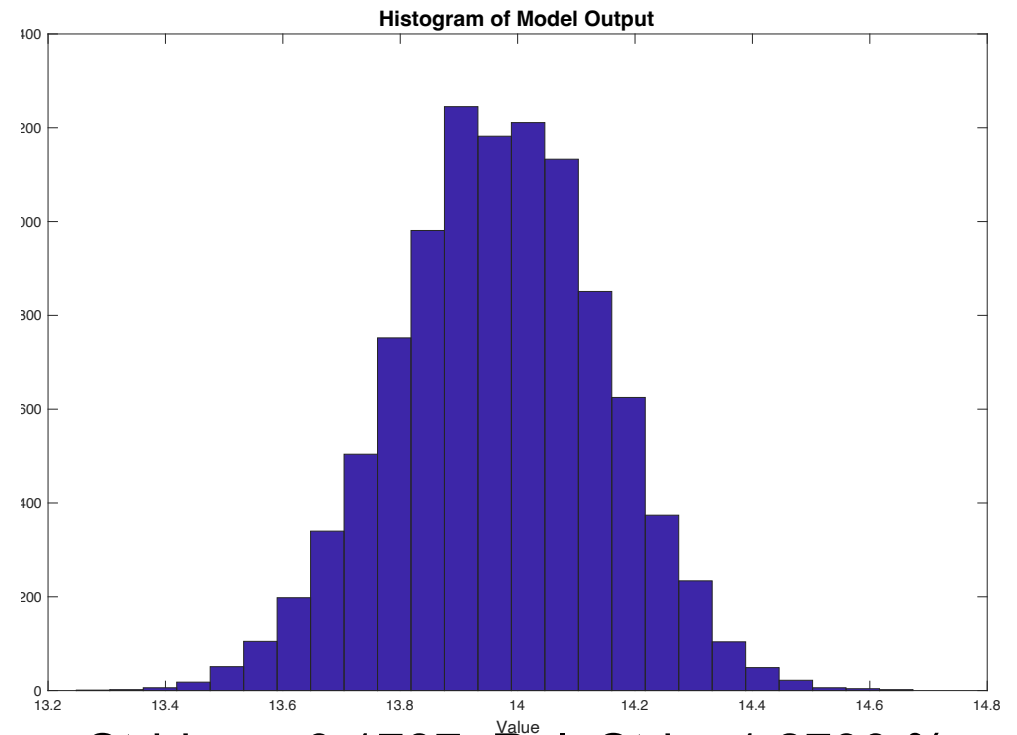
## Uncertainty Propagation: Creating the Output Realisations

### A squared log transformation of the answer to the universe, life and everything

...



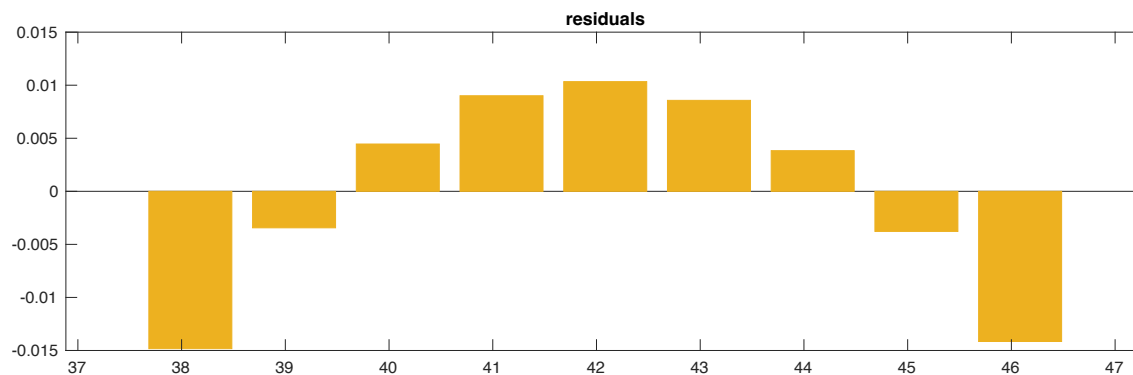
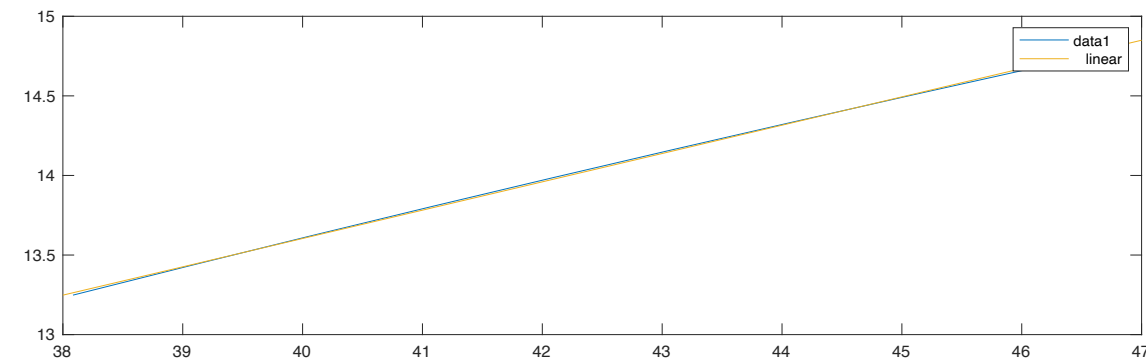
Mean = 13.9694



Stddev = 0.1787, Rel. Std = 1.2790 %



## Analytical Deduction of Uncertainty



$$c = (2 * \log(\mu) / \mu);$$

$$u2\_y = c^2 * \sigma^2;$$
$$u\_y = u2\_y^{(0.5)}$$

$$u\_y = 0.1780$$



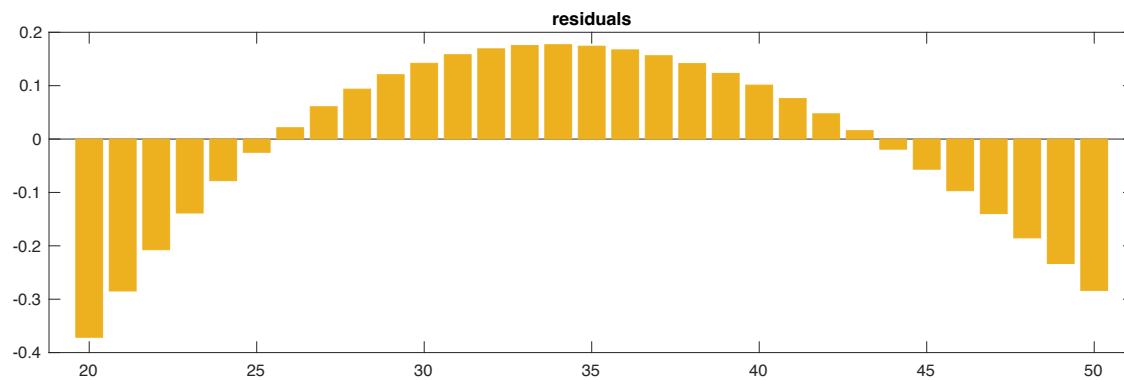
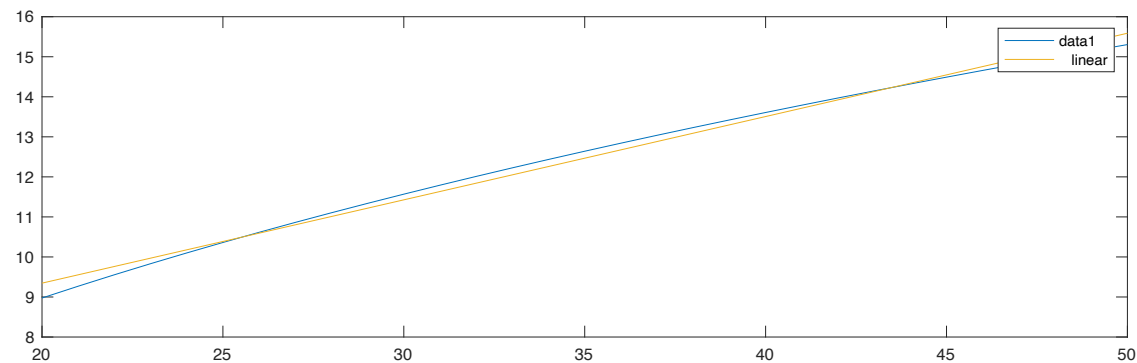
## Analytical Deduction of Uncertainty: Comparison with Monte Carlo

| N      | MC u(y) | Delta to Analytical |
|--------|---------|---------------------|
| 100    | 0.1540  | -0.0240             |
| 1000   | 0.1686  | -0.0094             |
| 10000  | 0.1763  | -0.0017             |
| 100000 | 0.1785  | 5.5467e-04          |

Analytical  $u(y) = 0.1780$



## Limits of Analytical Deduction of Uncertainty



Partial derivatives  
assume linearity over  
the range of values.



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measurements | products | policy

Come

# CoMet Toolkit

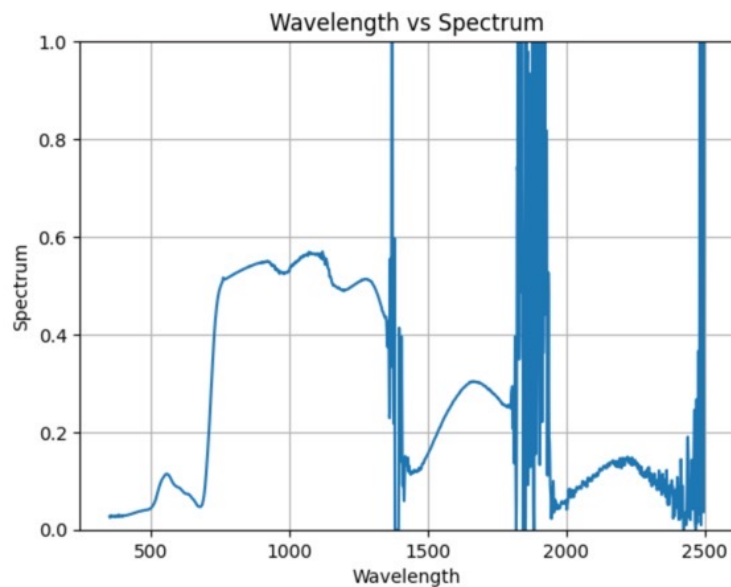


The **CoMet Toolkit** (Community Metrology Toolkit) is an open-source software project to develop Python tools for the handling of error-covariance information in the analysis of measurement data.

<https://www.comet-toolkit.org>



## A Simple Test: uncertainty propagated through the NDVI



Step 1: get some spectrum into Python

Blackfern\_INR ASD\_ASD FS FR-3\_BODY.csv  
Blackfern\_INR ASD\_ASD FS FR-3\_HDR.csv

```
# read file
file_path = 'Blackfern_INR ASD_ASD FS FR-3_BODY.csv'
spectral_data = read_numerical_csv(file_path)

# Extracting wavelength and spectrum data
wvl = np.array(spectral_data[0])
spectrum = np.array(spectral_data[1])
```



## Create a measurement function

```
# your measurement function
def getNDVI(wvl, spectrum):

    red_index = np.where(wvl == 650)[0]
    red = spectrum[red_index]

    nir_index = np.where(wvl == 750)[0]
    nir = spectrum[nir_index]

    ndvi = (nir - red) / (nir + red)

    return ndvi
```

And test it ...

```
NDVI=getNDVI(wvl, spectrum)

print(NDVI)
```



## Define your (assumed) uncertainty

# uncertainty of reflectance

`spectrum_ur = spectrum*0.02` # 2% random uncertainty, expressed as absolute uncertainty

`spectrum_us = np.ones(len(spectrum))*0.03` # systematic uncertainty of 3% (common between bands)



## Propagate through the measurement function

```
# uncertainty of NDVI
prop=punpy.MCPropagation(10000)

NDVI_ur=prop.propagate_random(getNDVI, [wv1, spectrum],
                               [np.zeros(len(wv1)), spectrum_ur])

NDVI_us=prop.propagate_systematic(getNDVI, [wv1, spectrum],
                                   [np.zeros(len(wv1)), spectrum_us])

NDVI_ut=(NDVI_ur**2+NDVI_us**2)**0.5 # add in quadrature

NDVI: [0.77499699]
NDVI_ur: 0.00564151
NDVI_us: 0.08720669
NDVI_ut: 0.08738898
```



## Interpretation

NDVI: 0.7749, uncertainty of 0.0873 at  $k=1$

Roughly speaking, we are 68% sure that the NDVI is between 0.689 and 0.860

In percent, that uncertainty of the NDVI is:

Relative combined uncertainty: 11.03 %



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**Questions?**



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**Original slides by Simon Trim follow hereafter**



## 7.2 Eight main steps to evaluating uncertainty

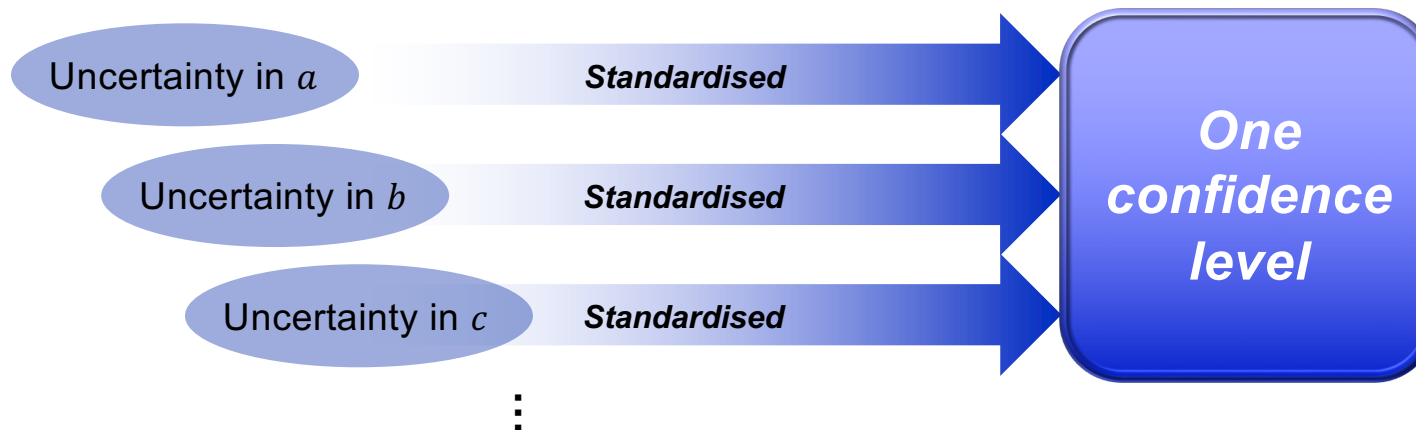
1. Decide what you need to find out from your measurements. Decide what actual measurements and calculations are needed to produce the final result.
2. Carry out the measurements needed.
3. Estimate the uncertainty of each input quantity that feeds into the final result. Express all uncertainties in similar terms.
4. Decide whether the errors of the input quantities are independent of each other. If you think not, then some extra calculations or information are needed.
5. Calculate the result of your measurement (including any known corrections for things such as calibration).
6. Find the combined standard uncertainty from all the individual aspects.
7. Express the uncertainty in terms of a coverage factor, together with a size of the uncertainty interval, and state a level of confidence.
8. Write down the measurement result and the uncertainty, and state how you got both of these.



## **8. Standard uncertainty for both Type A and Type B evaluations, combining standard uncertainties, correlation and coverage factor $k$**

## 8.1 Standard uncertainty

All contributing uncertainties should be expressed at the same confidence level, by converting them into standard uncertainties.

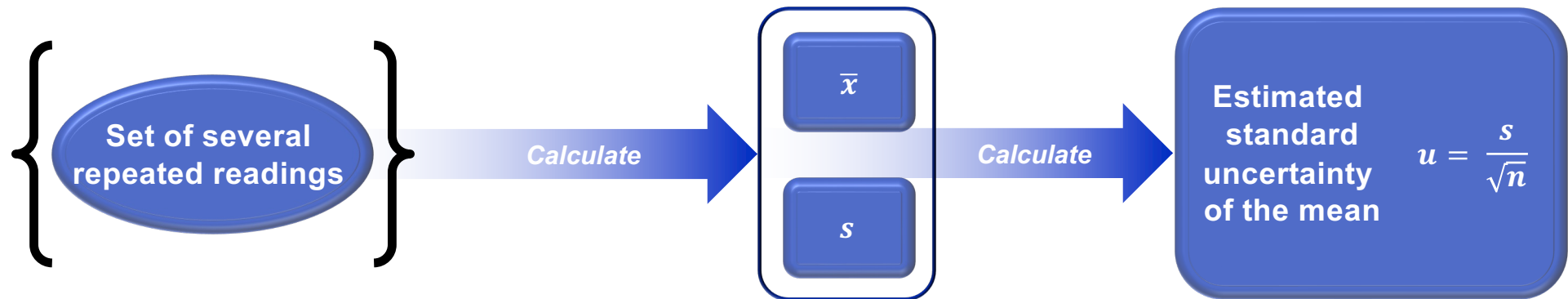


**Standard uncertainty**  $\equiv$  a **margin** whose size can be thought of as  $\pm$  **one** (estimated) **standard deviation** ( $\sigma$ ).

The standard uncertainty tells us about the *uncertainty of an average* (not just about the spread of values). A standard uncertainty is usually shown by the symbol  $u$ , or  $u(y)$  (*i.e.*, the standard uncertainty in  $y$ ).

## 8.1 Standard uncertainty

### Calculating standard uncertainty for a Type A evaluation



where  $n$  = number of measurements in the set.

*(The **standard uncertainty of the mean** has historically also been called the **standard deviation of the mean**, or the **standard error of the mean**.)*



## 8.1 Standard uncertainty

### Calculating standard uncertainty for a Type B evaluation

Where the information is more scarce (in some Type B estimates, for example when the measurement procedure cannot be repeated), you might only be able to estimate the upper and lower limits of uncertainty (*i.e.*, interval *a priori* known, for instance by means of calibration results).

Uncertainty may be evaluated from assumed probability distributions based on experience or other information.

The standard uncertainty for a uniform distribution is found from:

$$\frac{a}{\sqrt{3}}$$

where  $a$  is the semi-range (or half-width) between the upper and lower limits.

*N.B.: you can usually assume that uncertainties ‘imported’ from the calibration certificate for a measuring instrument are normally distributed.*

$$\bar{s} = \sqrt{\frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2}.$$



## 8.1 Standard uncertainty

### Converting uncertainties from one unit of measurement to another

Uncertainty contributions must be in the same units before they are combined. As the saying goes, you cannot 'compare apples with oranges'.

**Example:** in making a measurement of length  $L$ , the measurement uncertainty will also eventually be stated in terms of length.



- ⇒ One source of uncertainty might be the variation in room temperature  $T$ . Although the source of this uncertainty is temperature, the effect is in terms of length, and it must be accounted for in units of length.
- ⇒ E.g.: for every degree rise in temperature, the material being measured expands in length by 0.1 %.
- ⇒ For a piece of the material 100 *cm* long, and  $u(T) = \pm 2^\circ \text{C}$ , we have  $u(L) = \pm 0.2 \text{ cm}$ .



## 8.2 Combining standard uncertainties

Individual standard uncertainties calculated by Type A or Type B evaluations can be combined validly by 'summation in quadrature' (also known as 'root sum of the squares').

The result of this is called the **combined standard uncertainty**, shown by  $u_c$  or  $u_c(y)$ .

Summation in quadrature is simplest where the result of a measurement is reached by addition or subtraction.

The more complicated cases are also covered here.

## 8.2 Combining standard uncertainties

### Summation in quadrature for addition and subtraction

$$u_c = \sqrt{a^2 + b^2 + c^2 + \dots etc}$$

### Summation in quadrature for multiplication or division

You might need to find the area,  $A$ , of your cool rectangular pool, by multiplying the length,  $L$ , by the width,  $W$  (*i.e.*,  $A = L \times W$ ). The relative or fractional uncertainty in the area of the pool can be found from the fractional uncertainties in the length and width. For length  $L$  with uncertainty  $u(L)$ , the relative uncertainty is  $u(L)/L$ . For width  $W$ , the relative uncertainty is  $u(W)/W$ . Then the relative uncertainty  $u(A)/A$  in the area is given by:

$$\frac{u(A)}{A} = \sqrt{\left(\frac{u(L)}{L}\right)^2 + \left(\frac{u(W)}{W}\right)^2}$$



Just your average pool,  
nothing fancy.



## 8.2 Combining standard uncertainties

### Summation in quadrature for more complicated functions

⇒ Where a value is squared (e.g.  $Z^2$ ) in the calculation of the final measurement result, then the relative uncertainty due to the squared component is in the form:

$$\frac{2u(Z)}{Z}$$

⇒ Where a square root (e.g.  $\sqrt{Z}$ ) is part of the calculation of a result, then the relative uncertainty due to that component is in the form

$$\frac{u(Z)}{2Z}$$



## 8.2 Combining standard uncertainties

### Summation in quadrature for more complicated functions

Some measurements are processed using formulae which use combinations of addition, subtraction, multiplication and division, etc.

For example, you might measure electrical resistance  $R$  and voltage  $V$ , and then calculate the resulting power  $P$  using the relationship:

$$P = \frac{V^2}{R}$$

In this case, the relative uncertainty  $u(P)/P$  in the value of power would be given by:

$$\frac{u(P)}{P} = \sqrt{\left(\frac{2u(V)}{V}\right)^2 + \left(\frac{u(R)}{R}\right)^2}$$



## 8.3 Correlation

The equations given so far to calculate the combined standard uncertainty are only correct if the input standard uncertainties are not inter-related or correlated.

This means we usually need to question whether all the uncertainty contributions are independent.

Could a large error in one input cause a large error in another?

Could some outside influence, such as temperature, have a similar effect on several aspects of uncertainty at once - visibly or invisibly?

## 8.4 The law of propagation of uncertainties

To obtain the final uncertainty, uncertainties due to each element in the process that affect the final result must be combined – *i.e.*, they must be propagated through this process.

The GUM gives the Law of Propagation of Uncertainty:

$$u_c^2(y) = \underbrace{\sum_{i=1}^n \left( \frac{\partial f}{\partial x_i} \right)^2 u^2(x_i)}_{\text{Adding in quadrature}} + 2 \underbrace{\sum_{i=1}^{n-1} \sum_{j=i+1}^n \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} u(x_i, x_j)}_{\text{Covariance term}}$$

which applies for a measurement model of the form

$$Y = f(X_1, \dots, X_i, \dots, X_n)$$

where an estimate  $x_i$  of quantity  $X_i$  has an associated uncertainty  $u(x_i)$ . The quantity  $u_c^2(y)$  is the squared standard uncertainty (squared standard deviation of the probability distribution) associated with the measured value  $y$  which comes from a combination of the uncertainties associated with all the different effects,  $x_i$ .

## 8.4 The law of propagation of uncertainties

It can help to write the Law of Propagation of Uncertainty in terms of sensitivity coefficients as

$$u_c^2(y) = \underbrace{\sum_{i=1}^n c_i^2 u^2(x_i)}_{\text{Adding in quadrature}} + \underbrace{2 \sum_{i=1}^{n-1} \sum_{j=i+1}^n c_i c_j u(x_i, x_j)}_{\text{Covariance term}}$$

where the sensitivity coefficient is  $c_i = \partial f / \partial x_i$ .

The sensitivity coefficient is a ‘translation’ from one variable to another.

It answers the question: “how sensitive is  $y$  to an uncertainty associated with  $x_i$  ?”

In mathematical terms, the sensitivity coefficient is a *partial derivative* of the measurement function.



## 8.5 Sensitivity coefficients

There are 3 methods for determining sensitivity coefficients and they are all equally valid and all approved by the GUM:–



Mathematically (differentiating the measurement equations)



Numerically (modelling through an instrument model in software, or changing the input parameters to the measurement equation)



Experimentally (changing the effect in the lab and seeing how much the measured value changes)



## 8.6 Coverage factor $k$

Having scaled the components of uncertainty consistently, to find the combined standard uncertainty, we may then want to re-scale the result. The combined standard uncertainty may be thought of as equivalent to 'one standard deviation', but we may wish to have an overall uncertainty stated at another level of confidence, e.g. 95 %.

This re-scaling can be done using a coverage factor,  $k$ . Multiplying the combined standard uncertainty,  $u_c$ , by a coverage factor gives a result which is called the expanded uncertainty, usually shown by the symbol  $U$ , i.e.,

$$U = k u_c$$

Some coverage factors (for a normal distribution) are:

$k = 1$  for a confidence level of approximately 68 %

$k = 2.58$  for a confidence level of 95 %

$k = 3$  for a confidence level of 99.7 %

Other, less common, shapes of distribution have different coverage factors.

## 9. Exercise – Determining sensitivity coefficients through differentiation

Consider the radiance of a white diffuser panel illuminated by an FEL lamp. The radiance of the diffuser tile, viewed at an angle of 45° and for a normal incidence illumination, is given by

$$L_S = \frac{E_{FEL} \beta_{0^\circ:45^\circ}}{\pi} \frac{d_{cal}^2}{d_{use}^2}$$

where  $L_S$  is the source radiation,  $E_{FEL}$  is the lamp irradiance and  $\beta_{0^\circ:45^\circ}$  is the diffuser radiance factor. The calibration distance for the FEL lamp is  $d_{cal}$  and it is set at a distance  $d_{use}$  from the diffuser.



Calculate the sensitivity coefficient of the diffuser radiance due to the radiance of the lamp



$$c_{E_{FEL}} = \frac{\partial L_S}{\partial E_{FEL}} = \frac{\beta_{0^\circ:45^\circ}}{\pi} \frac{d_{cal}^2}{d_{use}^2} = \frac{L_S}{E_{FEL}}$$

## 9. Exercise – Determining sensitivity coefficients through differentiation

The absolute uncertainty (units [ $W m^{-2} sr^{-1} nm^{-1}$ ]) associated with the lamp diffuser radiance due to the lamp irradiance is the sensitivity coefficient times the uncertainty associated with the lamp irradiance (which has units [ $W m^{-2} nm^{-1}$ ]).

Write it.



$$u_{L_S:E_{FEL}} = c_{E_{FEL}} u_{E_{FEL}} = \frac{L_S}{E_{FEL}} u_{E_{FEL}}$$

$$\frac{u_{L_S:E_{FEL}}}{L_S} = \frac{u_{E_{FEL}}}{E_{FEL}}$$

*i.e.* the relative uncertainty associated with radiance of the lamp-diffuser (usually with uncertainty expressed in %) due to the lamp irradiance is equal to the relative uncertainty associated with the FEL irradiance (also expressed in %).



## 9. Exercise – Determining sensitivity coefficients through differentiation

*A similar relationship can be found for all the other parameters.*

Note that for the distance of use (the lamp to diffuser distance):

$$c_{d_{use}} = \frac{\partial L_S}{\partial d_{use}} = \frac{-2}{d_{use}^3} \frac{E_{FEL} \beta_{0^\circ:45^\circ}}{\pi} d_{cal}^2 = \frac{-2 L_S}{d_{use}}$$

Therefore

$$\frac{u_{L_S:d_{use}}}{L_S} = \frac{2u_{d_{use}}}{d_{use}}$$